Chapter 3
Introduction to Functional Equations

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Functional Equations: Introduction

Functional Equations:

- **Equations where the unknowns are functions**
- **Example.** Cauchy Equation:

$$\phi(x + y) = \phi(x) + \phi(y)$$

  ϕ is a solution of this equation if, for any two values x and y, the application of ϕ to $x + y$ equals the addition of the application of ϕ to x and to y.

- The equation establishes conditions that functions ϕ have to satisfy.
- **Example.** Typical solutions of this equation are the functions

$$\phi(x) = \alpha x$$

  for an arbitrary value for α.

Functional Equations: Introduction

- Uses of functional equations in information fusion:
  - when an aggregation operator is needed, we know which basic properties it has to satisfy.
    - First, conditions expressed as functional equations.
    - Second, the operator is derived from the equations.
  - to study the properties of a methods.
    - Because functional equations can characterize the operators.

      A characterization consists of finding a minimum set of properties (a minimum set of equations) that uniquely implies the operator.

      The set of properties that imply an operator is usually not unique.

Functional Equations: Introduction

- **Example.** (Theorem 3.1) Funcional equations for aggregation:
  $\rightarrow$ WM with nonrestricted weights
  - The most general function ϕ satisfying (for all x, y, t, u)

$$\phi(x+t, y+t) = \phi(x, y) + t \quad (1)$$

and

$$\phi(xu, yu) = \phi(x, y)u \text{ for } u \neq 0 \quad (2)$$

is

$$\phi(x, y) = (1-k)x + ky. \quad (3)$$

Functional Equations: Introduction

- This theorem can be seen
  - as a way to construct the function from the properties, or
  - as a characterization of Equation 3.
- The characterizations are not unique.
- **Example.** (Theorem 3.2) Another characterization of ϕ.
  - The most general function ϕ satisfying (for all x_1, x_2, y_1, y_2, x)

$$\phi(x_1 + y_1, x_2 + y_2) = \phi(x_1, x_2) + \phi(y_1, y_2)$$

and

$$\phi(x, x) = x$$

is

$$\phi(x, y) = (1 - k)x + ky.$$

Functional Equations: Basic equations I

- Some equations and their solutions:
  - First Cauchy equation (a continuous function $\phi : \mathbb{R} \to \mathbb{R}$):
    $$\phi(x + y) = \phi(x) + \phi(y)$$
    $\to$ solution: (for a real constant α) $\phi(x) = \alpha x$
  - Generalization of the Cauchy equation
    $$\phi(x_1 + y_1, x_2 + y_2, \ldots, x_N + y_N) =$$
    $$\phi(x_1, x_2, \ldots, x_N) + \phi(y_1, y_2, \ldots, y_N)$$
    $\to$ solution: (for an arbitrary real constant α_i)
    $\phi(x_1, x_2, \ldots, x_N) = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_N x_N$

Functional Equations: Basic equations II

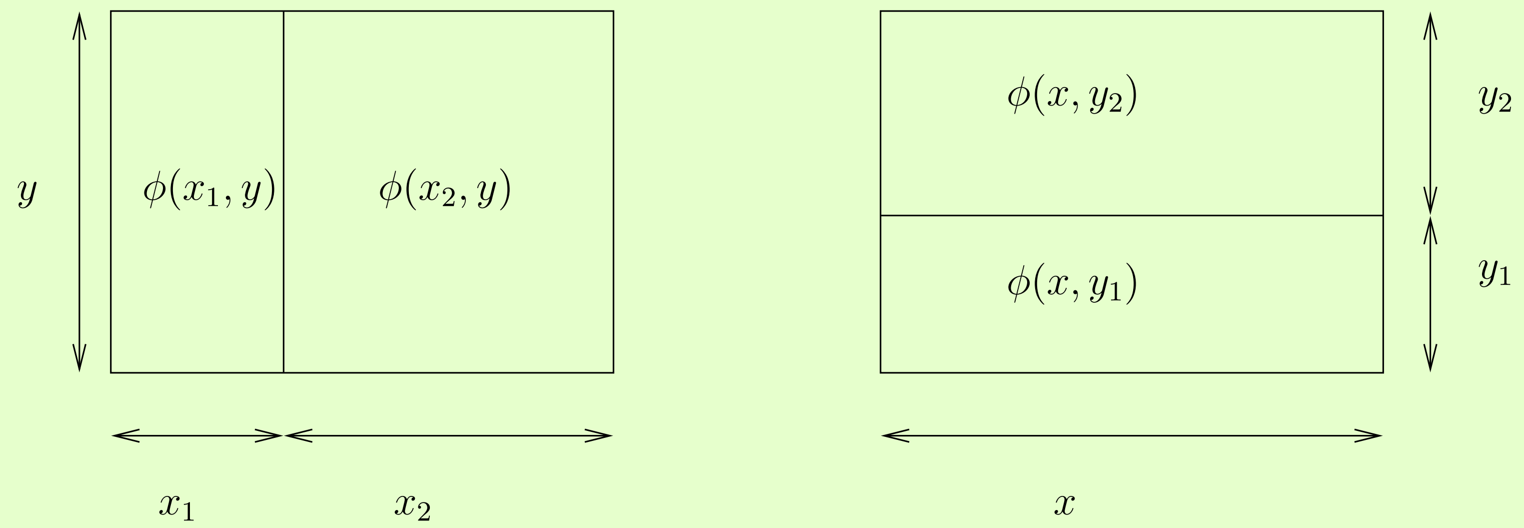
- More equations and their solutions:
  - Exponential equation
    $$\phi(x+y) = \phi(x)\phi(y)$$
    $\longrightarrow$ solution: (for an arbitrary real constant α) $\phi(x) = e^{\alpha x}$.
  - Logarithm equation (for all positive x and y)
    $$\phi(x \cdot y) = \phi(x) + \phi(y)$$
    $\longrightarrow$ solution: $\phi(x) = \alpha \log(x)$.
  - Equation (for all positive x, y)
    $$\phi(xy) = \phi(x)\phi(y)$$
    $\longrightarrow$ solution: $\phi(x) = x^c \quad \text{or} \quad \phi(x) = 0$

Functional Equations: Basic equations III

- Still more equations and their solutions:
  - N-term Jensen equation
    $$\phi(\frac{1}{N}\sum_{i=1}^{N} x_i) = \frac{1}{N}\sum_{i=1}^{N} \phi(x_i)$$
    $\rightarrow$ solution (α and β arbitrary real constants) $\phi(x) = \alpha x + \beta$
  - Equation (ϕ, ψ strictly monotone functions):
    $$\phi^{-1}(\frac{1}{N}\sum_{i=1}^{N} \phi(x_i)) = \psi^{-1}(\frac{1}{N}\sum_{i=1}^{N} \psi(x_i))$$
    $\rightarrow$ solution (α and β arbitrary real constants s.t. $\alpha \neq 0$): $\phi(x) = \alpha\psi(x) + \beta$
  - Equation
    $$\phi(\frac{x_1+y_1}{2}, \frac{x_2+y_2}{2}) = \frac{\phi(x_1,x_2)+\phi(y_1,y_2)}{2}$$
    $\rightarrow$ solution: $\phi(x, y) = \alpha x + \beta y + c$.

Functional Equations: Basic equations IV

- **Example.** Area of a rectangle $\phi(side_1, side_2)$
  - The most general positive solution of the system of equations
$$\phi(x_1 + x_2, y) = \phi(x_1, y) + \phi(x_2, y)$$
$$\phi(x, y_1 + y_2) = \phi(x, y_1) + \phi(x, y_2)$$
    is
$$\phi(x, y) = kxy.$$



Functional Equations: Basic equations IV

- **Example (proof).** Area of a rectangle $\phi(side_1, side_2)$
  - a) ϕ constant on y in Equation 1 and define $\psi(x) = \phi(x, y)$
    Eq. (1) $\rightarrow \psi(x_1 + x_2) = \psi(x_1) + \psi(x_2)$
    By, first Cauchy equation $\psi(x) = \alpha x$.
    Therefore, $\phi(x, y) = \alpha(y) \cdot x$ (because α depends on y).
  - (b_1) similarly, ... assume ϕ constant on x in second eq.
    Therefore, $\phi(x, y) = \beta(x) \cdot y$.
  - Thus, $\phi(x, y) = \alpha(y) \cdot x = \beta(x) \cdot y$.
  - Dividing by the product $x \cdot y$, we obtain
    $$\frac{\phi(x,y)}{x \cdot y} = \frac{\alpha(y)}{y} = \frac{\beta(x)}{x}.$$
  - The only way that $\alpha(y)/y$ is equal to $\beta(x)/x$ (for all x and y) is with both quotients always equal to a constant (k):
    $\frac{\phi(x,y)}{xy} = \frac{\alpha(y)}{y} = \frac{\beta(x)}{x} = k$.
  - Therefore, $\phi(x, y) = kxy$.

Functional Equations: Information Fusion

- s euros to m projects by N human experts.

	Proj 1	Proj 2	$\cdots$	Proj j	$\cdots$	Proj m
E_1	x_1^1	x_2^1	$\cdots$	x_j^1	$\cdots$	x_m^1
E_2	x_1^2	x_2^2	$\cdots$	x_j^2	$\cdots$	x_m^2
	$\vdots$	$\vdots$		$\vdots$		$\vdots$
E_i	x_1^i	x_2^i	$\cdots$	x_j^i	$\cdots$	x_m^i
	$\vdots$	$\vdots$		$\vdots$		$\vdots$
E_N	x_1^N	x_2^N	$\cdots$	x_j^N	$\cdots$	x_m^N
DM	$f_1(\mathbf{x_1})$	$f_2(\mathbf{x_2})$	$\cdots$	$f_j(\mathbf{x_j})$	$\cdots$	$f_m(\mathbf{x_m})$

Functional Equations: Information Fusion

- The general solution of the system (Proposition 3.11)

$$f_j : [0, s]^N \to \mathbb{R}^+ \text{ for } j = \{1, \cdots, m\} \tag{4}$$

$$\sum_{j=1}^{m} \mathbf{x}_j = \mathbf{s} \text{ implies that } \sum_{j=1}^{m} f_j(\mathbf{x}_j) = s \tag{5}$$

$$f_j(\mathbf{0}) = 0 \text{ for } j = 1, \cdots, m \tag{6}$$

for a given $m > 2$ is given by

$$f_1(\mathbf{x}) = f_2(\mathbf{x}) = \cdots = f_m(\mathbf{x}) = f((x_1, x_2, \ldots, x_N)) = \sum_{i=1}^{N} \alpha_i x_i,$$

where $\alpha_1, \cdots, \alpha_N$ are nonnegative constants satisfying $\sum_{i=1}^{N} \alpha_i = 1$, but are otherwise arbitrary.

Functional Equations: Solving Functional Equations

- Main techniques commonly used to solve functional equations
  - Variables by values (replace x by d_0)
  - Function transformation (replace a function by another)
  - Variable transformation (e.g. $x = e^u$)
  - Considering a more general equation
  - Variables as constants (e.g. $\psi(x) = \phi(x, y)$)
  - Separation of variables (e.g. as in the area of the rectangle)