Chapter 6
From the weighted mean to fuzzy integrals

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WM, OWA, and WOWA operators

- Operators
  - Weighting vector (dimension N): $v = (v_1 ... v_N)$ iff $v_i \in [0, 1]$ and $\sum_i v_i = 1$
  - Arithmetic mean (AM :$\mathbb{R}^N \to \mathbb{R}$): $AM(a_1, ..., a_N) = (1/N) \sum_{i=1}^N a_i$
  - Weighted mean (WM: $\mathbb{R}^N \to \mathbb{R}$): $WM_{\mathbf{p}}(a_1, ..., a_N) = \sum_{i=1}^N p_i a_i$ ($\mathbf{p}$ a weighting vector of dimension N)
  - Ordered Weighting Averaging operator (OWA: $\mathbb{R}^N \to \mathbb{R}$):

$$OWA_{\mathbf{w}}(a_1, ..., a_N) = \sum_{i=1}^{N} w_i a_{\sigma(i)},$$

where $\{\sigma(1), ..., \sigma(N)\}$ is a permutation of $\{1, ..., N\}$ s. t. $a_{\sigma(i-1)} \geq a_{\sigma(i)}$, and $\mathbf{w}$ a weighting vector.

WM, OWA, and WOWA operators

- Examples
  - Exam with three exercises.

    Different exercises with different weights:

    1st exercise 0.5, 2nd 0.25, 3rd 0.25

    $\longrightarrow$ situation modeled with a WM with $\mathbf{p} = (0.5,\ 0.25,\ 0.25)$.
  - Five judges in Olympic Games.

    Average of the rates of the judges disregarding extremes

    $\longrightarrow$ modeled with OWA with $\mathbf{w} = (0,\ 1/3,\ 1/3,\ 1/3,\ 0)$

WM, OWA, and WOWA operators

- Properties
  - $OWA_{(0,\ 0,\ ...,\ 0,\ 1)}(a_1, \ldots, a_N) = \min(a_1, \ldots, a_N)$
  - $OWA_{(1,\ 0,\ ...,\ 0,\ 0)}(a_1,\ \ldots,\ a_N) = \max(a_1,\ \ldots,\ a_N)$
  - Dictatorship: $WM_{\mathbf{p}}(a_1, \ldots, a_N) = a_i$
    when $p_i = 1$ and $p_j = 0$ for all $j \neq i$
  - OWA generalizes: order statistics (OS), median, kth minimum, kth maximum, AM, α-trimmed, (α, β)-trimmed means, α-winsorized, (α, β)-winsorized means

WM, OWA, and WOWA operators

- Median (M: $\mathbb{R}^N \to \mathbb{R}$):

$$M(a_1, \ldots, a_N) = \begin{cases} \frac{a_{\sigma(N/2)} + a_{\sigma(N/2+1)}}{2} & \text{when } N \text{is even} \\ a_{\sigma(\frac{N+1}{2})} & \text{when } N \text{is odd.} \end{cases}$$

  – When N is odd, $M = OS_{(N+1)/2}$.

- (r, s)-trimmed mean: AM after removing lowest/highest values

$$(a_{\sigma(r+1)} + \cdots + a_{\sigma(N-s)})/(N - r - s)$$

- (r, s)-winsorized means: AM where omitted values are replaced by the nearest ones

$$\frac{r \cdot a_{\sigma(r+1)} + a_{\sigma(r+1)} + \cdots + a_{\sigma(N-s)} + s \cdot x_{\sigma(N-s)}}{N}$$

WM, OWA, and WOWA operators

- Interpretation of weights in WM and OWA. Scenarios:
  - Multicriteria Decision Making
    Several criteria are used to evaluate several alternative
  - Fuzzy Constraint Satisfaction Problems
    Given a possible solution, constraints are evaluated in [0,1] (0: not satisfied at all, 1: completely satisfied, (0,1): partial satisfaction)
  - Robot Sensing (all data, same time instant)
    Sensors measuring the distance to the same object
  - Robot Sensing (all data, different time instants)
    Sensors measuring the distance to the same object

WM, OWA, and WOWA operators

- Interpretation of weights in WM and OWA (i.e., p and w)
  - Multicriteria Decision Making.
    p: importance of criteria,
    w: degree of compensation
  - Fuzzy Constraint Satisfaction Problems.
    p: importance of the constraints,
    w: degree of compensation
  - Robot Sensing (all data, same time instant).
    p: reliability of each sensor,
    w: importance of small values/outliers
  - Robot Sensing (all data, different time instants).
    p: more importance to recent data than old one,
    w: importance of small values/outliers

WM, OWA, and WOWA operators

Example. A and B teaching a tutorial+training course w/ constraints

- The total number of sessions is six.
- Professor A will give the tutorial, which should consist of about three sessions; three is the optimal number of sessions; a difference in the number of sessions greater than two is unacceptable.
- Professor B will give the training part, consisting of about two sessions.
- Both professors should give more or less the same number of sessions. A difference of one or two is half acceptable; a difference of three is unacceptable.

WM, OWA, and WOWA operators

Example. Formalization

- Variables
  - x_A: Number of sessions taught by Professor A
  - x_B: Number of sessions taught by Professor B
- Constraints
  - the constraints are translated into
    - C_1: $x_A + x_B$ should be about 6
    - C_2: x_A should be about 3
    - C_3: x_B should be about 2
    - C_4: $|x_A - x_B|$ should be about 0
  - using fuzzy sets, the constraints are described ...

WM, OWA, and WOWA operators

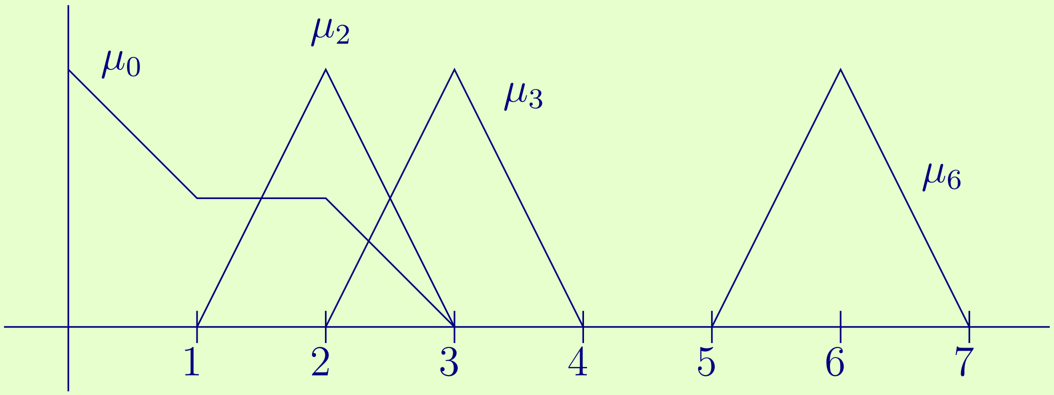
Example. Formalization

- Constraints
  - if fuzzy set μ_6 expresses "about 6," then,
    we evaluate "$x_A + x_B$ should be about 6" by $\mu_6(x_A + x_B)$.
    $\longrightarrow$ given μ_6, μ_3, μ_2, μ_0,
  - Then, given a solution pair (x_A, x_B), the degrees of satisfaction:
    * $\mu_6(x_A + x_B)$
    * $\mu_3(x_A)$
    * $\mu_2(x_B)$
    * $\mu_0(|x_A - x_B|)$

WM, OWA, and WOWA operators

Example. Formalization

- Membership functions for constraints



WM, OWA, and WOWA operators

Example. Application

alternative	Satisfaction degrees	Satisfaction degrees			
(x_A, x_B)	$(\mu_6(x_A + x_B),\ \mu_3(x_A),\ \mu_2(x_B),\ \mu_0(\lvert x_A - x_B \rvert))$	C_1	C_2	C_3	C_4
$(2, 2)$	$(\mu_6(4),\ \mu_3(2),\ \mu_2(2),\ \mu_0(0))$	0	0.5	1	1
$(2, 3)$	$(\mu_6(5),\ \mu_3(2),\ \mu_2(3),\ \mu_0(1))$	0.5	0.5	0.5	0.5
$(2, 4)$	$(\mu_6(6),\ \mu_3(2),\ \mu_2(4),\ \mu_0(2))$	1	0.5	0	0.5
$(3.5, 2.5)$	$(\mu_6(6),\ \mu_3(3.5),\ \mu_2(2.5),\ \mu_0(1))$	1	0.5	0.5	0.5
$(3, 2)$	$(\mu_6(5),\ \mu_3(3),\ \mu_2(2),\ \mu_0(1))$	0.5	1	1	0.5
$(3, 3)$	$(\mu_6(6),\ \mu_3(3),\ \mu_2(3),\ \mu_0(0))$	1	1	0.5	1

WM, OWA, and WOWA operators

Example. Application

- Let us consider the following situation:
  - Professor A is more important than Professor B
  - The number of sessions equal to six is the most important constraint (not a *crisp* requirement)
  - The difference in the number of sessions taught by the two professors is the least important constraint

WM with $\mathbf{p} = (p_1, p_2, p_3, p_4) = (0.5, 0.3, 0.15, 0.05)$.

WM, OWA, and WOWA operators

Example. Application

- WM with $\mathbf{p} = (p_1, p_2, p_3, p_4) = (0.5, 0.3, 0.15, 0.05)$.

alternative	Aggregation of the Satisfaction degrees	WM
(x_A, x_B)	$WM_{\mathbf{p}}(C_1, C_2, C_3, C_4)$	
$(2, 2)$	$WM_{\mathbf{p}}(0, 0.5, 1, 1)$	0.35
$(2, 3)$	$WM_{\mathbf{p}}(0.5, 0.5, 0.5, 0.5)$	0.5
$(2, 4)$	$WM_{\mathbf{p}}(1, 0.5, 0, 0.5)$	0.675
$(3.5, 2.5)$	$WM_{\mathbf{p}}(1, 0.5, 0.5, 0.5)$	0.75
$(3, 2)$	$WM_{\mathbf{p}}(0.5, 1, 1, 0.5)$	0.725
$(3, 3)$	$WM_{\mathbf{p}}(1, 1, 0.5, 1)$	0.925

WM, OWA, and WOWA operators

Example. Application

- Compensation: how many values can have a bad evaluation
- One bad value does not matter: OWA with $\mathbf{w} = (1/3, 1/3, 1/3, 0)$ (lowest value discarded)

alternative	Aggregation of the Satisfaction degrees	OWA
(x_A, x_B)	$OWA_{\mathbf{w}}(C_1, C_2, C_3, C_4)$	
$(2, 2)$	$OWA_{\mathbf{w}}(0, 0.5, 1, 1)$	0.8333
$(2, 3)$	$OWA_{\mathbf{w}}(0.5, 0.5, 0.5, 0.5)$	0.5
$(2, 4)$	$OWA_{\mathbf{w}}(1, 0.5, 0, 0.5)$	0.6666
$(3.5, 2.5)$	$OWA_{\mathbf{w}}(1, 0.5, 0.5, 0.5)$	0.6666
$(3, 2)$	$OWA_{\mathbf{w}}(0.5, 1, 1, 0.5)$	0.8333
$(3, 3)$	$OWA_{\mathbf{w}}(1, 1, 0.5, 1)$	1.0

WM, OWA, and WOWA operators

- Weighted Ordered Weighted Averaging WOWA operator (WOWA :$\mathbb{R}^N \to \mathbb{R}$):

$$WOWA_{\mathbf{p},\mathbf{w}}(a_1, ..., a_N) = \sum_{i=1}^{N} \omega_i a_{\sigma(i)}$$

where

$$\omega_i = w^*(\sum_{j \leq i} p_{\sigma(j)}) - w^*(\sum_{j < i} p_{\sigma(j)}),$$

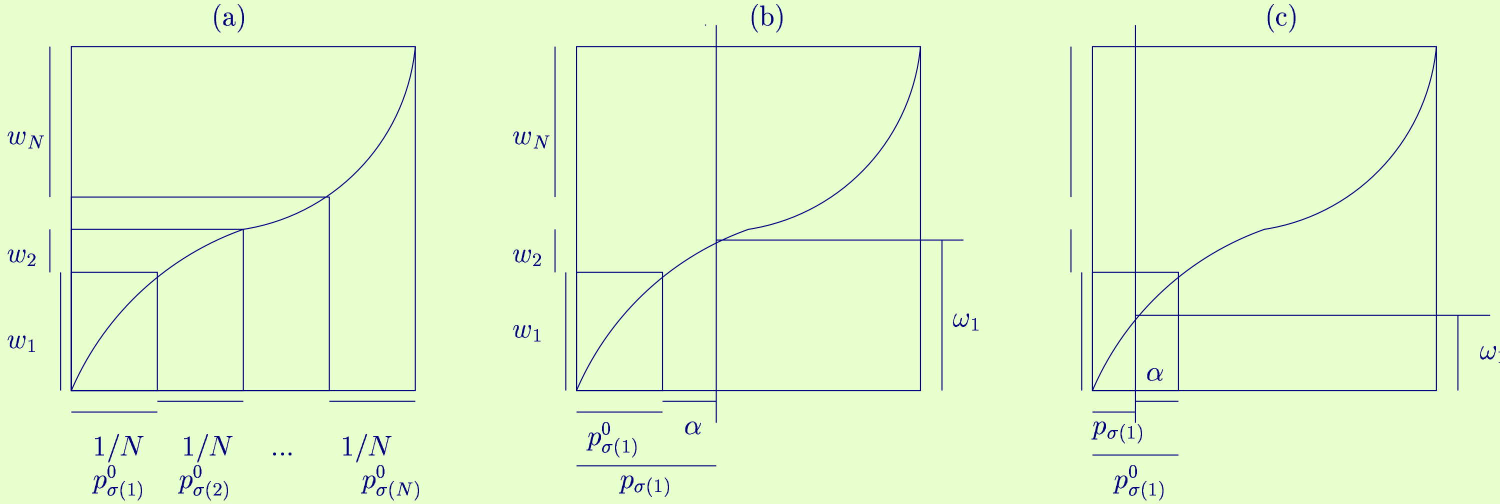
with σ a permutation of $\{1, ..., N\}$ s. t. $a_{\sigma(i-1)} \geq a_{\sigma(i)}$, and w^* a nondecreasing function that interpolates the points

$$\{(i/N, \sum_{j \leq i} w_j)\}_{i=1,...,N} \cup \{(0,0)\}.$$

w^* is required to be a straight line when the points can be interpolated in this way.

WM, OWA, and WOWA operators

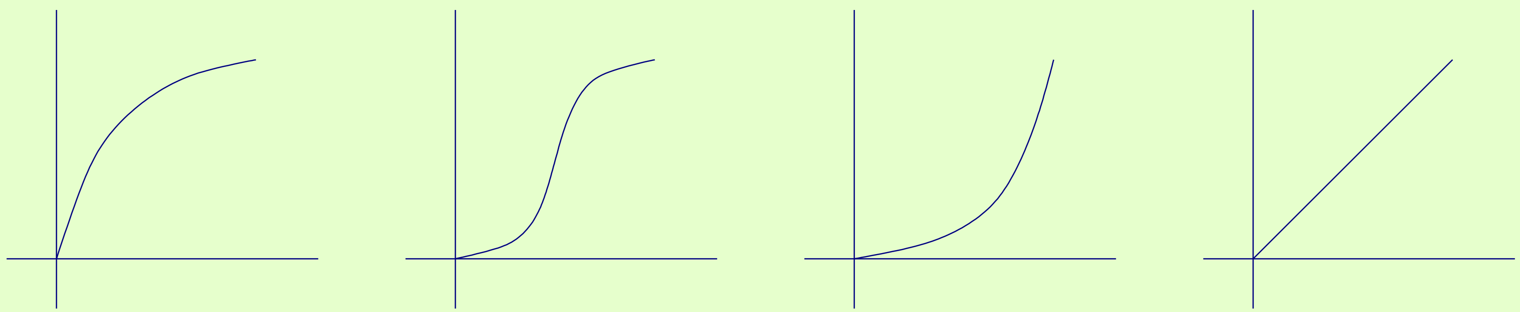
- Construction of the w^* quantifier



- Rationale for new weights (ω_i, for each value a_i) in terms of $\mathbf{p}$ and $\mathbf{w}$.
  - If a_i is small, and small values have more importance than larger ones, increase p_i for a_i (i.e., $\omega_i \geq p_{\sigma(i)}$).
    (the same holds if the value a_i is large and importance is given to large values)
  - If a_i is small, and importance is for large values, $\omega_i < p_{\sigma(i)}$
    (the same holds if a_i is large and importance is given to small values).

WM, OWA, and WOWA operators

- The shape of the function w^* gives importance
  - (a) to large values
  - (b) to medium values
  - (c) to small values
  - (d) equal importance to all values



(a) (b) (c) (d)

WM, OWA, and WOWA operators

Example. Application

- Importance for constraints as given above: $\mathbf{p} = (0.5, 0.3, 0.15, 0.05)$
- Compensation as given above: $\mathbf{w} = (1/3, 1/3, 1/3, 0)$ (lowest value discarded)
  $\rightarrow$ WOWA with $\mathbf{p}$ and $\mathbf{w}$.

alternative	Aggregation of the Satisfaction degrees	WOWA
(x_A, x_B)	$WOWA_{\mathbf{p},\mathbf{w}}(C_1, C_2, C_3, C_4)$	
$(2, 2)$	$WOWA_{\mathbf{p},\mathbf{w}}(0, 0.5, 1, 1)$	0.4666
$(2, 3)$	$WOWA_{\mathbf{p},\mathbf{w}}(0.5, 0.5, 0.5, 0.5)$	0.5
$(2, 4)$	$WOWA_{\mathbf{p},\mathbf{w}}(1, 0.5, 0, 0.5)$	0.8333
$(3.5, 2.5)$	$WOWA_{\mathbf{p},\mathbf{w}}(1, 0.5, 0.5, 0.5)$	0.8333
$(3, 2)$	$WOWA_{\mathbf{p},\mathbf{w}}(0.5, 1, 1, 0.5)$	0.8
$(3, 3)$	$WOWA_{\mathbf{p},\mathbf{w}}(1, 1, 0.5, 1)$	1.0

WM, OWA, and WOWA operators

- Properties
  - The WOWA operator generalizes the WM and the OWA operator.
    - When $\mathbf{p} = (1/N \ \ldots \ 1/N)$, OWA

$$WOWA_{\mathbf{p},\mathbf{w}}(a_1, ..., a_N) = OWA_{\mathbf{w}}(a_1, ..., a_N) \text{ for all } \mathbf{w} \text{ and } a_i.$$

    - When $\mathbf{w} = (1/N \ ... \ 1/N)$, WM

$$WOWA_{\mathbf{p},\mathbf{w}}(a_1, ..., a_N) = WM_{\mathbf{p}}(a_1, ..., a_N) \text{ for all } \mathbf{p} \text{ and } a_i.$$

    - When $\mathbf{w} = \mathbf{p} = (1/N \ ... \ 1/N)$, AM

$$WOWA_{\mathbf{p},\mathbf{w}}(a_1, ..., a_N) = AM(a_1, ..., a_N)$$

Choquet integrals

- In WM, we combine a_i w.r.t. weights p_i.
  $\longrightarrow$ a_i is the value supplied by information source x_i.
  Formally
  - $X = \{x_1, \ldots, x_N\}$ is the set of information sources
  - $f : X \to \mathbb{R}^+$ the values supplied by the sources
    $\longrightarrow$ then $a_i = f(x_i)$

  Thus,

$$WM_{\mathbf{p}}(a_1, ..., a_N) = \sum_{i=1}^{N} p_i a_i = \sum_{i=1}^{N} p_i f(x_i) = WM_{\mathbf{p}}(f(x_1), ..., f(x_N))$$

Choquet integrals

- In WM, a single weight for each element.
  That is, $p_i = p(x_i)$ (where, x_i is the source supplying a_i)
  $\rightarrow$ when we consider a set $A \subset X$, *weight* of A???

  ... fuzzy measures $\mu(A)$

  Formally,

  – Fuzzy measure ($\mu : \wp(X) \rightarrow [0,1]$), a set function satisfying
  (i) $\mu(\emptyset) = 0$, $\mu(X) = 1$ (boundary conditions)
  (ii) $A \subseteq B$ implies $\mu(A) \leq \mu(B)$ (monotonicity)

Choquet integrals

- Choquet integral of f w.r.t. μ (alternative notation, $CI_\mu(a_1, \ldots, a_N)/CI_\mu(f)$)

$$(C)\int f d\mu = \sum_{i=1}^{N} [f(x_{s(i)}) - f(x_{s(i-1)})]\mu(A_{s(i)}),$$

where s in $f(x_{s(i)})$ is a permutation so that $f(x_{s(i-1)}) \leq f(x_{s(i)})$ for $i \geq 1$, $f(x_{s(0)}) = 0$, and $A_{s(k)} = \{x_{s(j)} | j \geq k\}$ and $A_{s(N+1)} = \emptyset$.

- Alternative expressions (Proposition 6.18):

$$(C)\int f d\mu = \sum_{i=1}^{N} f(x_{\sigma(i)})[\mu(A_{\sigma(i)}) - \mu(A_{\sigma(i-1)})],$$

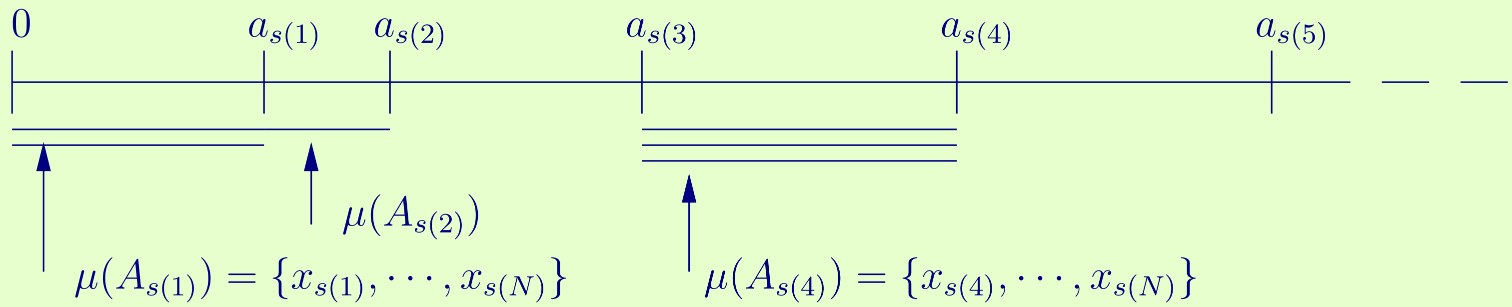
$$(C)\int f d\mu = \sum_{i=1}^{N} f(x_{s(i)})[\mu(A_{s(i)}) - \mu(A_{s(i+1)})],$$

where σ is a permutation of $\{1, \ldots, N\}$ s.t. $f(x_{\sigma(i-1)}) \geq f(x_{\sigma(i)})$, where $A_{\sigma(k)} = \{x_{\sigma(j)} | j \leq k\}$ for $k \geq 1$ and $A_{\sigma(0)} = \emptyset$

Choquet integrals

- Different equations point out different aspects of the CI

(6.1) $(C)\int f d\mu = \sum_{i=1}^{N} [f(x_{s(i)}) - f(x_{s(i-1)})]\mu(A_{s(i)})$,

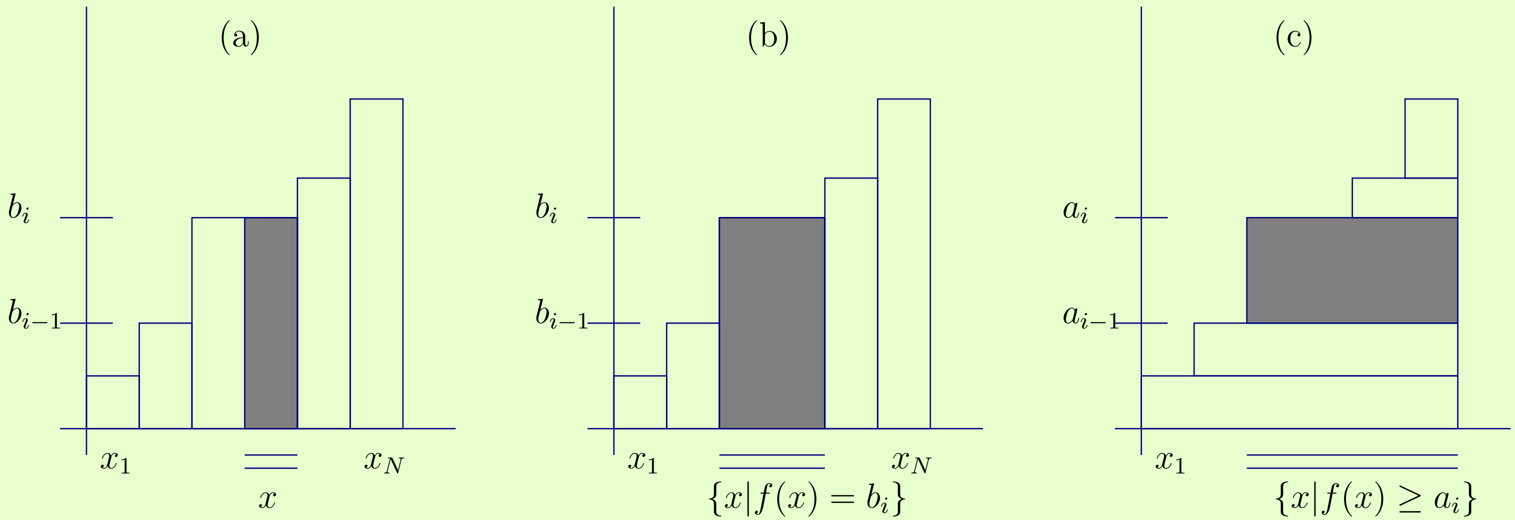


(6.2) $(C)\int f d\mu = \sum_{i=1}^{N} f(x_{\sigma(i)})[\mu(A_{\sigma(i)}) - \mu(A_{\sigma(i-1)})]$,

Choquet integrals

- $\int f d\mu =$ (for additive measures)

  (6.5) $\sum_{x \in X} f(x)\mu(\{x\})$
  (6.6) $\sum_{i=1}^{R} b_i \mu(\{x | f(x) = b_i\})$
  (6.7) $\sum_{i=1}^{N} (a_i - a_{i-1})\mu(\{x | f(x) \geq a_i\})$
  (6.8) $\sum_{i=1}^{N} (a_i - a_{i-1})\big(1 - \mu(\{x | f(x) \leq a_{i-1}\})\big)$

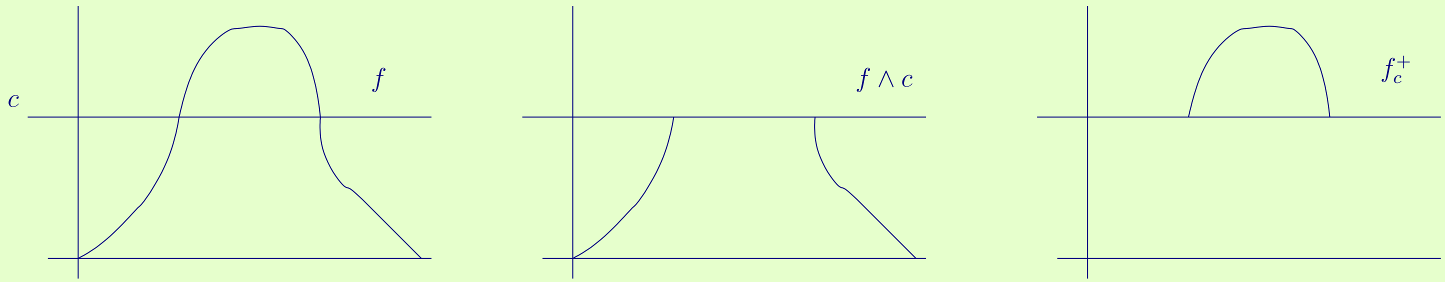


- Among (6.5), (6.6) and (6.7), only (6.7) satisfies internality.

Choquet integrals

- Properties of CI
  - Horizontal additive because $CI_\mu(f) = CI_\mu(f \wedge c) + CI_\mu(f_c^+)$
    ($f = (f \wedge c) + f_c^+$ is a horizontal additive decomposition of f)
    where, f_c^+ is defined by (for $c \in [0,1]$)

$$f_c^+ = \begin{cases} 0 & \text{if } f(x) \leq c \\ f(x) - c & \text{if } f(x) > c. \end{cases}$$



Choquet integrals

- Definitions (X a reference set, f, g functions $f, g : X \to [0,1]$)
  - $f < g$ when, for all x_i,
  $$f(x_i) < g(x_i)$$
  - f and g are comonotonic if, for all $x_i, x_j \in X$,
  $$f(x_i) < f(x_j) \text{ imply that } g(x_i) \leq g(x_j)$$
  - $\mathbb{C}$ is comonotonic monotone if and only if, for comonotonic f and g,
  $$f \leq g \text{ imply that } \mathbb{C}(f) \leq \mathbb{C}(g)$$
  - $\mathbb{C}$ is comonotonic additive if and only if, for comonotonic f and g,
  $$\mathbb{C}(f+g) = \mathbb{C}(f) + \mathbb{C}(g)$$

- Characterization. Let $\mathbb{C}$ satisfy the following properties
  - $\mathbb{C}$ is comonotonic monotone
  - $\mathbb{C}$ is comonotonic additive
  - $\mathbb{C}(1, \ldots, 1) = 1$

  Then, there exists μ s.t. $\mathbb{C}(f)$ is the CI of f w.r.t. μ.

Choquet integrals

- Properties
  - WM, OWA and WOWA are particular cases of CI.
    * WM with weighting vector $\mathbf{p}$ is a CI w.r.t. $\mu_{\mathbf{p}}(B) = \sum_{x_i \in B} p_i$
    * OWA with weighting vector $\mathbf{w}$ is a CI w.r.t. $\mu_{\mathbf{w}}(B) = \sum_{i=1}^{|B|} w_i$
    * WOWA with w.v. $\mathbf{p}$ and $\mathbf{w}$ is a CI w.r.t. $\mu_{\mathbf{p},\mathbf{w}}(B) = w^*(\sum_{x_i \in B} p_i)$
  - Any symmetric CI is an OWA operator.
  - Any CI with a distorted probability is a WOWA operator.
  - Let A be a crisp subset of X; then, the Choquet integral of A with respect to μ is $\mu(A)$.

    Here, the integral of A corresponds to the integral of its characteristic function, or, in other words, to the integral of the function f_A defined as $f_A(x) = 1$ if and only if $x \in A$.

Weighted Minimum and Weighted Maximum

- Possibilistic weighting vector (dimension N): $\mathbf{v} = (v_1 ... v_N)$ iff $v_i \in [0, 1]$ and $\max_i v_i = 1$.

- Weighted minimum (WMin: $[0, 1]^N \to [0, 1]$):
$WMin_{\mathbf{u}}(a_1, ..., a_N) = \min_i \max(neg(u_i), a_i)$
(alternative definition can be given with $\mathbf{v} = (v_1, \ldots, v_N)$ where $v_i = neg(u_i)$)

- Weighted maximum (WMax: $[0, 1]^N \to [0, 1]$):
$WMax_{\mathbf{u}}(a_1, ..., a_N) = \max_i \min(u_i, a_i)$

Weighted Minimum and Weighted Maximum

Example 6.34. Application of WMin and WMax to the tutorial example $\rightarrow$ possibilistic weights are given

Example 6.35. In a fuzzy inference

R_i: **IF** x is A_i **THEN** y is B_i.

- with disjunctive rules, the (fuzzy) output for a particular y_0 is a WMax

$$\tilde{B}(y_0) = \vee_{i=1}^{N} \big(B_i(y_0) \wedge A_i(x_0)\big).$$

- with conjunctive rules, and Kleene-Dienes implication ($\mathcal{I}(x,y) = \max(1-x, y)$) the (fuzzy) output of the system for a particular y_0 is a WMin

$$\tilde{B}(y_0) = \wedge_{i=1}^{N} \big(\mathcal{I}(A_i(x_0), B_i(y_0))\big) = \wedge_{i=1}^{N} \max(1 - A_i(x_0), B_i(y_0)).$$

that with $\mathbf{u} = (A_1(x_0), \ldots, A_N(x_0))$

$$\tilde{B}(y_0) = WMin_{\mathbf{u}}(B_1(y_0), \ldots, B_N(y_0)).$$

Weighted Minimum and Weighted Maximum

- Only operators in ordinal scales (max, min, neg) are used in $WMax$ and $WMin$.
- neg is completely determined in an ordinal scale

Proposition 6.36. Let $L = \{l_0, \ldots, l_r\}$ with $l_0 <_L l_1 <_L \cdots <_L l_r$; then, there exists only one function, $neg : L \to L$, satisfying

(N1) if $x <_L x'$ then $neg(x) >_L neg(x')$ for all x, x' in L.
(N2) $neg(neg(x)) = x$ for all x in L.

This function is defined by $neg(x_i) = x_{r-i}$ for all x_i in L

- Properties. For $\mathbf{u} = (1, \ldots, 1)$
  - $WMIN_{\mathbf{u}} = \min$
  - $WMAX_{\mathbf{u}} = \max$

Sugeno integral

- Sugeno integral of f w.r.t. μ (alternative notation, $SI_\mu(a_1, \ldots, a_N)/SI_\mu(f)$)

$$(S)\int f d\mu = \max_{i=1,N} \min(f(x_{s(i)}), \mu(A_{s(i)})),$$

  where s in $f(x_{s(i)})$ is a permutation so that $f(x_{s(i-1)}) \leq f(x_{s(i)})$ for $i \geq 2$, and $A_{s(k)} = \{x_{s(j)} | j \geq k\}$.

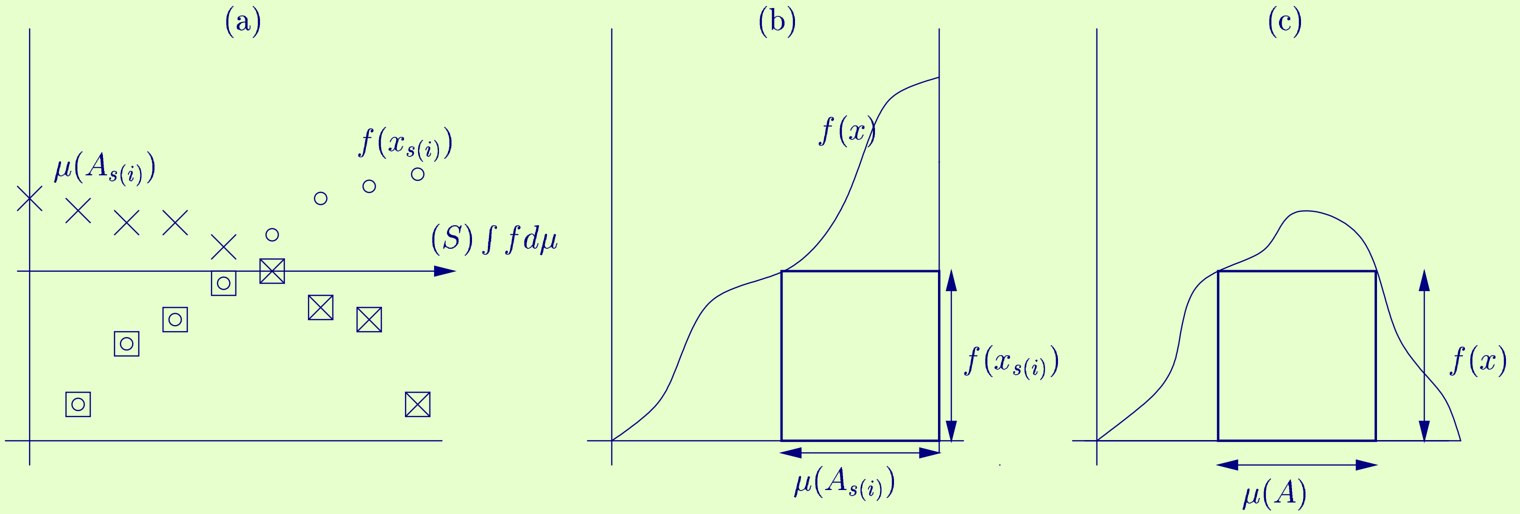
- Alternative expression (Proposition 6.38):

$$\max_i \min(f(x_{\sigma(i)}), \mu(A_{\sigma(i)})),$$

  where σ is a permutation of $\{1, \ldots, N\}$ s.t. $f(x_{\sigma(i-1)}) \geq f(x_{\sigma(i)})$, where $A_{\sigma(k)} = \{x_{\sigma(j)} | j \leq k\}$ for $k \geq 1$

Sugeno integral

- Graphical interpretation of Sugeno integrals



Sugeno integral

- Properties
  - WMin and WMax are particular cases of SI
    - WMax with weighting vector $\mathbf{u}$ is a SI w.r.t. $\mu_{\mathbf{u}}^{wmax}(A) = \max_{a_i \in A} u_i$.
    - WMin with weighting vector $\mathbf{u}$ is a SI w.r.t. $\mu_{\mathbf{u}}^{wmin}(A) = 1 - \max_{a_i \notin A} u_i$.

Fuzzy integrals

- Fuzzy integrals that generalize Choquet and Sugeno integrals
  - The fuzzy t-conorm integral
  - The twofold integral

Fuzzy integrals

- The fuzzy t-conorm integral
  - *The space of values of integrands* (F): The domain is denoted by $D = [0, 1]$, and the function to integrate is such that $f : X \to D$. The corresponding t-conorm is denoted by Δ. So, $F = (D, \Delta)$.
  - *The space of values of measures* (M): The domain is denoted by $T = [0, 1]$, thus, $\mu : \wp(X) \to T$. The corresponding t-conorm is $\perp$. Therefore, $M = (T, \perp)$.
  - *The space of values of integrals* (I): The domain is denoted by $\bar{T} = [0, 1]$, and the corresponding t-conorm is $\underline{\perp}$. Thus, $I = (\bar{T}, \underline{\perp})$.

$$\underbrace{\sum_{i=1,N} \underbrace{\left(a_{s(i)} - a_{s(i-1)}\right)}_{F} \underbrace{\mu(A_{s(i)})}_{M}}_{I}$$

$$\underbrace{\sum_{i=1,N} \underbrace{\underbrace{\left(a_{s(i)} - a_{s(i-1)}\right)}_{F} \underbrace{\mu(A_{s(i)})}_{M}}_{I^*}}_{I}$$

Fuzzy integrals

- The fuzzy t-conorm integral
  - $\mathcal{F} = (\Delta, \perp, \underline{\perp}, \otimes)$ is a *t-conorm system for integration* iff
    1. $\Delta, \perp$, and $\underline{\perp}$, are continuous t-conorms that are the maximum or Archimedean.
    2. $\otimes : ([0,1], \Delta) \times ([0,1], \perp) \rightarrow ([0,1], \underline{\perp})$ is a product-like operation fulfilling
       (a) $\otimes$ is continuous on $(0,1]^2$
       (b) $a \otimes x = 0$ if and only if $a = 0$ or $x = 0$
       (c) when $x \perp y < 1$, $a \otimes (x \perp y) = (a \otimes x) \underline{\perp} (a \otimes y)$ for all $a \in [0,1]$
       (d) when $a \Delta b < 1$, $(a \Delta b) \otimes x = (a \otimes x) \underline{\perp} (b \otimes x)$ for all $x \in [0,1]$.

Fuzzy integrals

- The fuzzy t-conorm integral
  - Given a t-conorm $\perp$, the substraction operator $-_\perp$ (Definition 2.51) is defined by:

$$x -_\perp y := inf\{z | y \perp z \geq x\}.$$

Example

(i) If $\perp$ is an Archimedean t-conorm with generator g, then

$$x -_\perp y = g^{(-1)}(g(x) - g(y)).$$

(ii) If $\perp$ is equal to the maximum, then

$$x -_{\max} y = \begin{cases} x & \text{if } x \geq y \\ 0 & \text{if } x < y. \end{cases}$$

Fuzzy integrals

- Fuzzy t-conorm integral of f based on $(\Delta, \perp, \underline{\perp}, \otimes)$ w.r.t. μ

$$(\mathcal{F}) \int f \otimes d\mu = \underline{\perp}_{i=1}^{N} (a_i -_{\Delta} a_{i-1}) \otimes \mu(A_{s(i)}),$$

where $a_i = f(x_{s(i)})$ with $f(x_{s(i)}) \leq f(x_{s(i+1)})$ and $a_0 = f(x_{s(0)}) = 0$, and $A_{s(k)} = \{x_{s(j)} | j \geq k\}$.

- Properties
  - When the t-conorm system is $(\hat{+}$[1]$, \hat{+}, \hat{+}, \cdot)$, we have CI.
  - When the t-conorm system is $(\max, \max, \max, \min)$, we have a SI.

[1] $\hat{+}$ denotes the bounded sum (Example 2.48)

Fuzzy integrals

- Twofold integral of f w.r.t. two fuzzy measures μ_C and μ_S
  - μ_S has a "fuzzy flavor", corresponds to the fuzzy measure of the SI
  - μ_C has a "probabilistic flavor", corresponds to the one of the CI

$$TI_{\mu_S,\mu_C}(f) = \sum_{i=1}^{N} \Big(\big(\bigvee_{j=1}^{i} f(x_{s(j)}) \wedge \mu_S(A_{s(j)}) \big) \big(\mu_C(A_{s(i)}) - \mu_C(A_{s(i+1)}) \big) \Big)$$

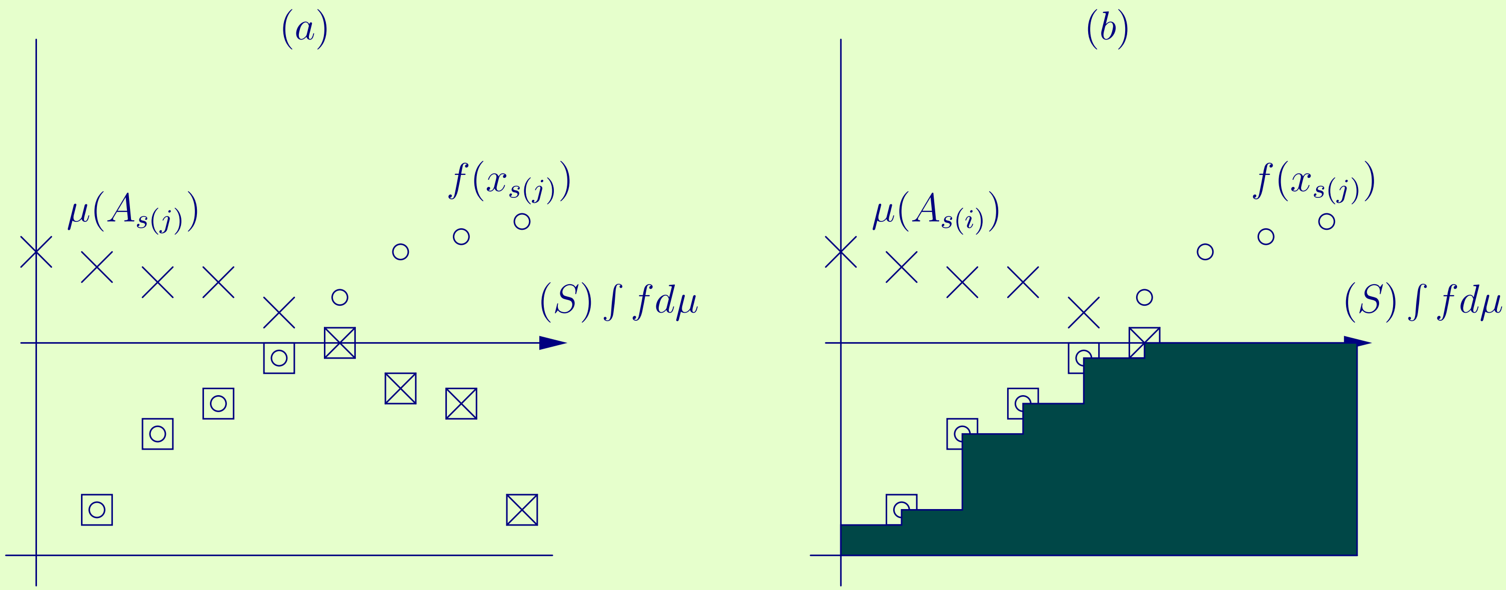
where s in $f(x_{s(i)})$ is a permutation so that $f(x_{s(i-1)}) \leq f(x_{s(i)})$ for $i \geq 1$, and $A_{s(k)} = \{x_{s(j)} | j \geq k\}$ and $A_{s(N+1)} = \emptyset$.

Fuzzy integrals

- Properties
  - When $\mu_C = \mu^*$, the TI reduces to the SI:
    $TI_{\mu_S,\mu_C}(a_1, \ldots, a_n) = SI_{\mu_S}(a_1, \ldots, a_n)$
  - When $\mu_S = \mu^*$, the TI reduces to the CI:
    $TI_{\mu_S,\mu_C}(a_1, \ldots, a_n) = CI_{\mu_C}(a_1, \ldots, a_n)$
  - When $\mu_C = \mu_S = \mu^*$, the TI reduces to the maximum:
    $TI_{\mu_S,\mu_C}(a_1, \ldots, a_n) = \bigvee(a_1, \ldots, a_n)$

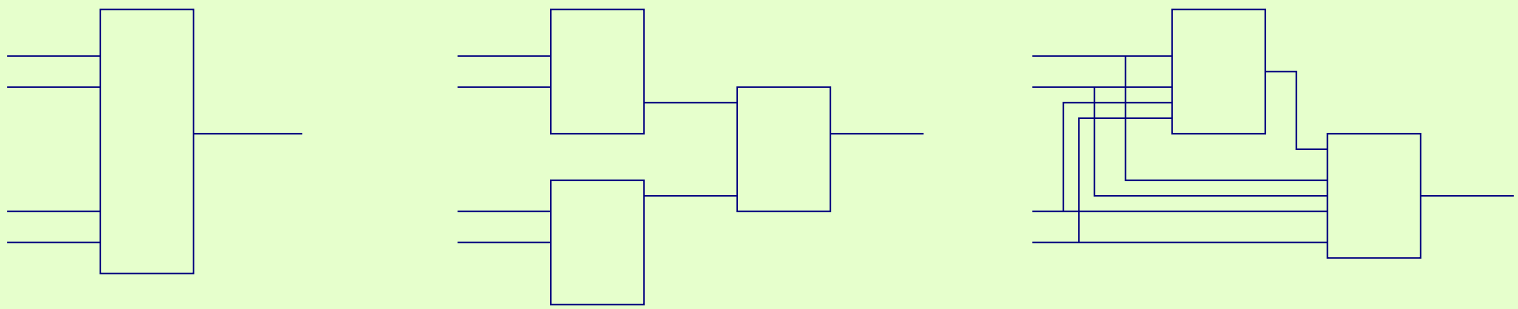
Fuzzy integrals

- Properties and graphical interpretation
  - $TI_{\mu_S,\mu_C}(f) \leq CI_{\mu_C}(f)$
  - $TI_{\mu_S,\mu_C}(f) \leq SI_{\mu_S}(f)$
  - $TI_{\mu_S,\mu_C}(f) = CI_{\mu_C}\big(f \wedge SI_{\mu_S}(f)\big)$



Hierarchical Models for Aggregation

- Hierarchical model



- Properties. The following conditions hold

  (i) Every multistep Choquet integral is a monotone increasing, positively homogeneous, piecewise linear function.

  (ii) Every monotone increasing, positively homogeneous, piecewise linear function on a full-dimensional convex set in $\mathbb{R}^N$ is representable as a two-step Choquet integral such that the fuzzy measures of the first step are additive and the fuzzy measure of the second step is a 0-1 fuzzy measure.