Chapter 7
Indices and Evaluation Methods

April 25, 2007

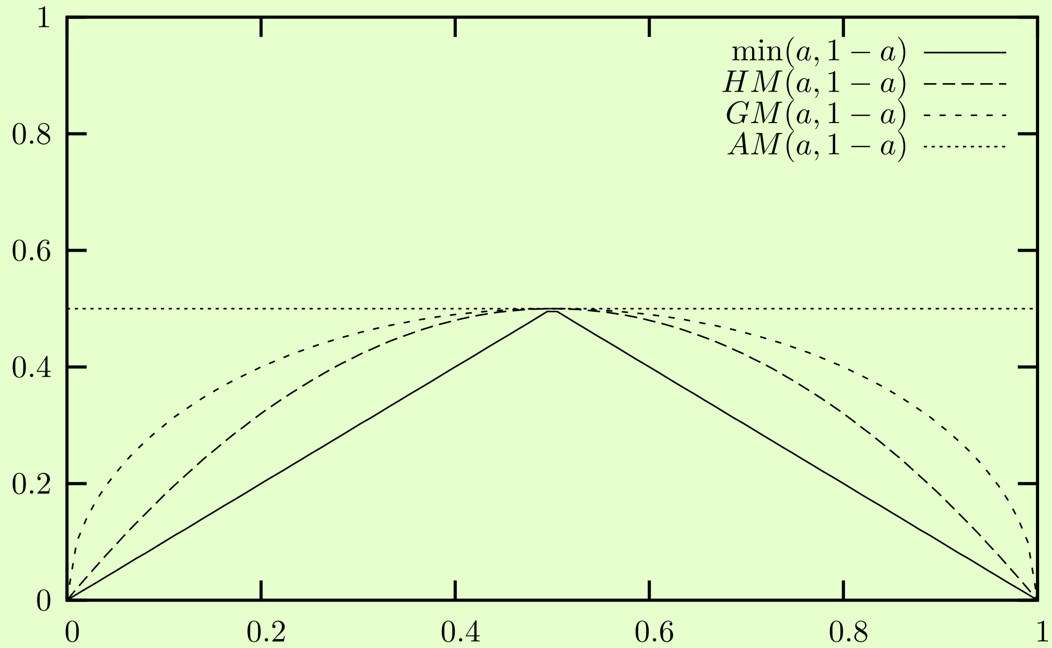
Vicenç Torra, Yasuo Narukawa (2007) Modeling Decisions: Information Fusion and Aggregation Operators, Springer. http://www.springer.com/3-540-68789-0; http://www.mdai.cat/ifao

Introduction

- Evaluation of the aggregation methods and their parameters
  - Graphical representations
  - Indices for fuzzy measures (fuzzy games)
    - Shapley, Banzhaf
  - Interaction index
  - Dispersion (entropy) for weights (but also for fuzzy measures)
  - Degree of disjunction
  - Tools in robust statistics
    - Influence function, gross-error sensitivity, local-shift sensitivity

Graphical representation

- Representation of $\mathcal{C}_{\mathbb{C}}(x) = \mathbb{C}(x, neg(x))$:
  - $\mathcal{C}_{AM}(x) = (x + 1 - x)/2 = 1/2$
  - $\mathcal{C}_{GM}(x) = \sqrt{x(1-x)}$
  - $\mathcal{C}_{HM}(x) = 2(1-x)x$



Indices of power: Shapley and Banzhaf

- 0-1 fuzzy measures on X can be used to model coalitions on X (games)
  $\mu(A) = 1$ is A is a winning coalition
  - Indices to model the power of x in X
- Interest in information fusion because for crisp sets:

$$CI_{\mu}(\chi_A) = \mu(A)$$

$$SI_{\mu}(\chi_A) = \mu(A)$$

  χ_A the characteristic function of the set A

- Different indices, different ways of defining the power

Indices of power: Shapley and Banzhaf

- Shapley-Shubik of μ for x_i (μ a 0-1 measure)
  - $\varphi_{x_i}(\mu)$ counts the number of times that x_i changes a losing coalition into a winning one.
    I.e., $\mu(S \cup \{x_i\}) = 1$ while $\mu(S) = 0$
  - For fuzzy measures (Shapley index) measures the difference.
    I.e., $\mu(S \cup \{x_i\}) - \mu(S)$
  - This is achieved considering all orderings ρ_X ($X!$ different orders):

$$\varphi_{x_i}(\mu) = \frac{1}{N!} \sum_{r \in \rho_X} \left(\mu(r_{x_i} \cup \{x_i\}) - \mu(r_{x_i})\right)$$

    there are different alternative expressions
  - There are characterizations of this value (see Theorem 7.5)

Indices of power: Shapley and Banzhaf

- Banzhaf Value of μ for x_i
  - Shapley counts twice some sets. E.g.,
    - $X = \{x_1, x_2, x_3\}$ s.t. $\mu(\{x_1, x_2\}) = 0$ and $\mu(\{x_1, x_2, x_3\}) = 1$.
    - The Shapley value for x_3 counts twice the fact that x_3 changes $\mu(\{x_1, x_2\})$, equal to 0 to $\mu(\{x_1, x_2, x_3\})$ equal to 1.
      $\rightarrow$ This is so because both orderings, $r_1 = (x_1, x_2, x_3)$ and $r_2 = (x_2, x_1, x_3)$, will be considered when computing $\varphi_{x_3}(\mu)$.
  - This is solved considering the pairs S and $S \setminus \{x_i\}$:
    unnormalized (or *nonstandardized* or *absolute*) *Banzhaf index*:

$$\beta'_{x_i}(\mu) := \frac{\sum_{S \subseteq X} \left(\mu(S) - \mu(S \setminus \{x_i\})\right)}{2^{N-1}}.$$

Average values

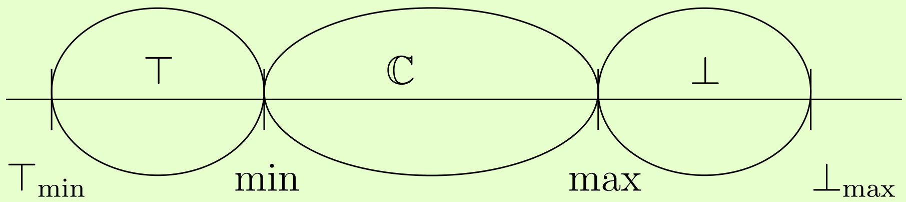
- Average value of $\mathbb{C}$ (function on $[0,1]^N$ with parameter P):

$$AV(\mathbb{C}_P) := \int_0^1 \dots \int_0^1 C_P(a_1, \dots, a_N) \, da_1 \dots da_N.$$

  - $AV(\min) = N/(N+1)$
  - $AV(\max) = 1/(N+1)$
  - $AV(AM) = 1/2$

Orness or the Degree of Disjunction

- t-norms (conjunction), binary aggregation operators, and t-conorms (disjunction):



- Orness of $\mathbb{C}_P$: similarity to the maximum:

$$\textit{orness}(\mathbb{C}_P) := \frac{AV(\mathbb{C}_P) - AV(\min)}{AV(\max) - AV(\min)}$$

- Andness: $andness = 1 - orness$

Orness or the Degree of Disjunction

- A few orness:
  - $orness(\max) = 1$
  - $orness(\min) = 0$
  - $orness(AM) = 1/2$
  - $orness(WM_{\mathbf{p}}) = 1/2$
  - $orness(GM_{\mathbf{p}=(p_1,\ldots,p_N)}) = \frac{N+1}{N-1}\left(\frac{N}{N+1}\right)^N - \frac{1}{N-1}$
  - $orness(OWA_{\mathbf{w}}) = \frac{1}{N-1}\sum_{i=1}^N (N-i)w_i$
  - $orness(HM_{\mathbf{p}=(p_1,p_2)}) = 0.2274$; $orness(HM_{\mathbf{p}=(p_1,p_2,p_3)}) = 0.2257$
  - $orness(CI_\mu) = \frac{1}{N-1}\sum_{A \subseteq X} \frac{N-|A|}{|A|+1} m(A)$ (m: Definition 5.14)

- Properties:
  - $orness(GM)_{\mathbf{p}=(p_1,\ldots,p_N)} < orness(GM)_{\mathbf{p}=(p_1,\ldots,p_{N+1})}$.
    E.g., for $N = 2$, $orness(GM) = 1/3 = 0.3333$
    for $N = 3$ $orness(GM) = 11/32 = 0.3437$.

Orness or the Degree of Disjunction

- Continuous Orness for a Fuzzy Quantifiers Q (OWA)
  - Definition:

$$orness(Q) := \int_0^1 Q(x)\; dx.$$

  - Examples of quantifiers

    Sugeno λ-quantifier: $(\lambda > -1)$
    * if $\lambda = 0$, $Q_\lambda(x) = x$
    * if $\lambda \neq 0$, $Q_\lambda(x) = (e^{x \ln(1+\lambda)} - 1)/\lambda$.

    Yager α-quantifier: $(\alpha > 0)$

$$Q_\alpha(x) = x^\alpha.$$

Orness or the Degree of Disjunction

- Continuous Orness for a Fuzzy Quantifiers Q (OWA)
  - Orness of the Sugeno λ-quantifier: $(\lambda = 0,\ orness(Q_\lambda) = 1/2)$

$$orness(Q_\lambda) = \frac{1}{ln(1+\lambda)} - \frac{1}{\lambda}.$$

  - Orness of the Yager α-quantifier:

$$orness(Q_\alpha) = \frac{1}{\alpha + 1}.$$

Orness or the Degree of Disjunction

- Orness of the Q_λ, $\lambda \in (-1, 100]$ (left) and of Q_α, $\alpha \in (0, 100]$