83rd EWG-MCDA 2016

Choquet integral: distributions and decisions

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Overview

Basics and objectives:

- Distribution based on the Choquet integral (for non-additive measures)

Motivation:

- Theory: Mathematical properties
- Methodology: different ways to express interactions
- Application: Decision (MCDM), classification, statistical disclosure control (data privacy)

Outline

1. Preliminaries
2. Choquet integral based distribution
3. Choquet-Mahalanobis based distribution
4. Summary

Preliminaries
Aggregation operators and the Choquet integral in Decision

MCDM: Aggregation for (numerical) utility functions

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- Decision, utility functions

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  Decision making process:

  Modelling=Criteria + Utilities, aggregation, selection

	Number of seats	Security	Price	Confort	trunk
Ford T	0	20	0	20	0
Seat 600	60	0	100	0	50
Simca 1000	100	30	100	50	70
VW Beetle	80	50	30	70	100
Citroën Acadiane	20	40	60	40	0

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  Decision making process:

  Modelling, aggregation = $\mathbb{C}$, selection

	Seats	Security	Price	Comfort	trunk	$\mathbb{C} = AM$
Ford T	0	20	0	20	0	8
Seat 600	60	0	100	0	50	42
Simca 1000	100	30	100	50	70	70
VW	80	50	30	70	100	66
Citr. Acadiane	20	40	60	40	0	32

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Aggregation operators are appropriate because they satisfy monotonicity

- Pareto dominance: Given two vectors $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$, we say that b dominates a when $a_i \leq b_i$ for all i and there is at least one k such that $a_k < b_k$.

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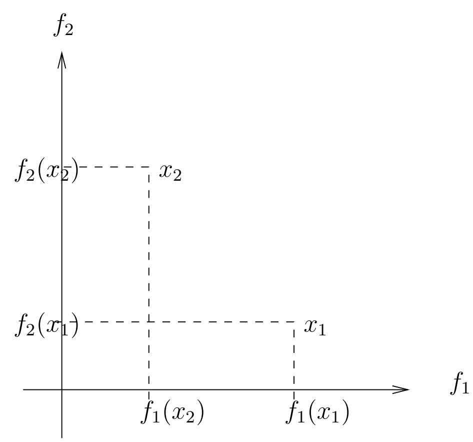
- Each one wins at least in one criteria to another one

Aggregation and Choquet integral in MCDM

- Pareto set, Pareto frontier, or non dominance set:
Given a set of alternatives U represented by vectors $u = (u_1, \ldots, u_n)$, the Pareto frontier is the set $u \in U$ such that there is no other $v \in U$ such that v dominates u.

$$PF = \{u | \text{there is no } v \text{ s.t. } v \text{ dominates } u\}$$

- Pareto optimal: an element u of the Pareto set



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- Decision making process:

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- The function of aggregation functions
  - Different aggregations lead to different orders (in the PF)
  - Aggregation establishes which points are *equivalent*
  - Different aggregations, lead to different curves of points (level curves)

Criteria
Satisfaction on:

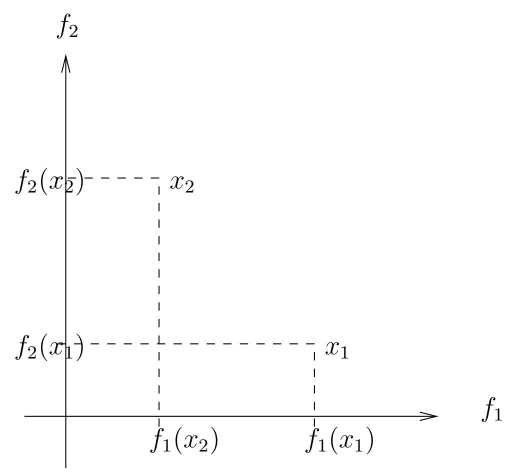
alt	Price	Quality	Comfort
FordT	0.2	0.8	0.3
206	0.7	0.7	0.8
...	...		

→

alt	Consensus
FordT	0.35
206	0.72
...	...

→

alt	Ranking
206	0.72
FordT	0.35
...	...



Aggregation and Choquet integral in MCDM

- Aggregation functions and different level curves
  - Arithmetic mean
  - Geometric mean, Harmonic mean, ...
  - Weighted mean
  - OWA, ...

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- Aggregation functions and different level curves
  - Arithmetic mean
  - Geometric mean, Harmonic mean, ...
  - Weighted mean
  - OWA, ...
  - Choquet integral (generalization of the AM, WM, OWA)
    - to represent interactions between criteria
    - non-independent criteria allowed

Aggregation and Choquet integral in MCDM

- Aggregation functions and parameters
  - Arithmetic mean: no parameters
  - Geometric mean, Harmonic mean, ...: : no parameters
  - Weighted mean: weighting vector
  - OWA, ...: weighting vector

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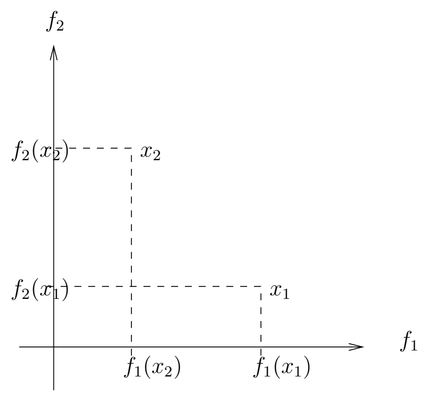
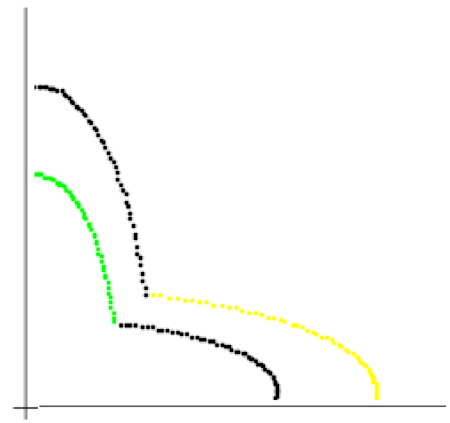
  - We can, of course, use *w(security,price,confort)=w(security)+w(price)+w(confort)*
  - but also
    - *w(security,price,confort)* $>$ *w(security)+w(price)+w(confort)*

    or
    - *w(security,price,confort)* $<$ *w(security)+w(price)+w(confort)*

Aggregation and Choquet integral in MCDM

- Aggregation functions and parameters
  - Choquet integral (generalization of the AM, WM, OWA): a measure
    - ∗ And the level curves ? decision ?





Preliminaries
Non-additive (fuzzy) measures and the Choquet integral

Definitions: measures

Additive measures.

- $(X, \mathcal{A})$ a measurable space; then, a set function μ is an additive measure if it satisfies
  (i) $\mu(A) \geq 0$ for all $A \in \mathcal{A}$,
  (ii) $\mu(X) \leq \infty$
  (iii) Finite case:
  $\mu(A \cup B) = \mu(A) + \mu(B)$ for disjoint A, B

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  $\mu(A \cup B) = \mu(A) + \mu(B)$ for disjoint A, B
- Probability and weights: $\mu(X) = 1$

Definitions: measures

Non-additive (or fuzzy) measures.

- $(X, \mathcal{A})$ a measurable space, a non-additive measure μ on $(X, \mathcal{A})$ is a set function $\mu : \mathcal{A} \to [0, 1]$ satisfying the following axioms:

(i) $\mu(\emptyset) = 0$

(ii) $\mu(X) \leq \infty$

(iii) $A \subseteq B$ implies $\mu(A) \leq \mu(B)$ (monotonicity)

- Weights: $\mu(X) = 1$

Definitions: measures

Non-additive measures. Examples. Distorted probabilities

- $m : \mathbb{R}^+ \to \mathbb{R}^+$ a continuous and increasing function such that $m(0) = 0$; P be a probability.
  The following set function μ_m is a non-additive measure:

$$\mu_{m,P}(A) = m(P(A)) \quad (1)$$

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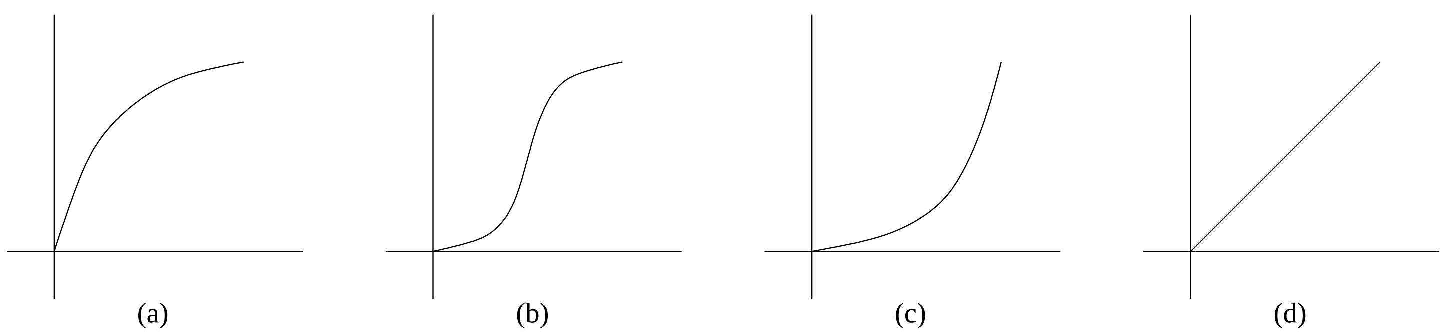
- If $m(x) = x^2$, then $\mu_m(A) = (P(A))^2$
- If $m(x) = x^p$, then $\mu_m(A) = (P(A))^p$



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Applications.

- To represent interactions

Definitions: integrals

Choquet integral (Choquet, 1954):

- μ a non-additive measure, g a measurable function. The Choquet integral of g w.r.t. μ, where $\mu_g(r) := \mu(\{x|g(x) > r\})$:

$$(C)\int g d\mu := \int_0^\infty \mu_g(r) dr. \tag{3}$$

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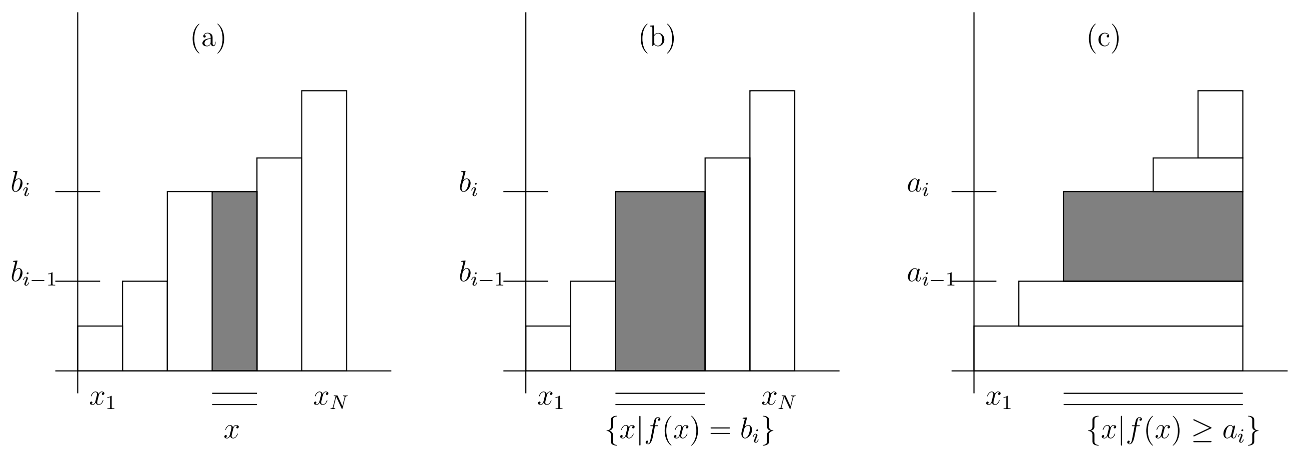


Definitions: integrals

Choquet integral. Discrete version

- μ a non-additive measure, f a measurable function. The Choquet integral of f w.r.t. μ,

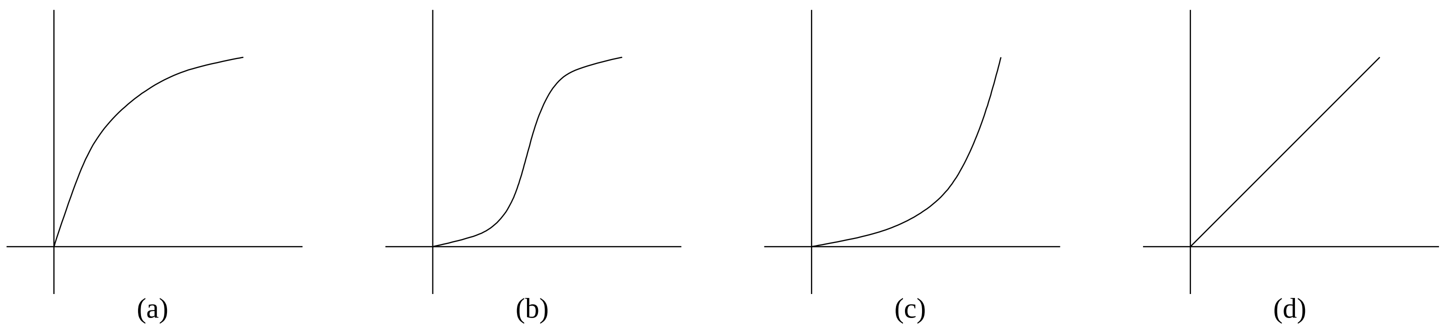
$$(C)\int f d\mu = \sum_{i=1}^{N}[f(x_{s(i)}) - f(x_{s(i-1)})]\mu(A_{s(i)}),$$

where $f(x_{s(i)})$ indicates that the indices have been permuted so that $0 \leq f(x_{s(1)}) \leq \cdots \leq f(x_{s(N)}) \leq 1$, and where $f(x_{s(0)}) = 0$ and $A_{s(i)} = \{x_{s(i)}, \ldots, x_{s(N)}\}$.

Definitions: measures

Choquet integral: Properties:

- When μ is additive, CI corresponds to the weighted mean
- CI can represent min, max, mean, order statistics, ...
- When μ is $\mu_{m,P}(A) = m(P(A))$ with $m(x) = x^p$,
  $CI_{\mu_m}(f)$
  $(a) \rightarrow$ *max*, $(b) \rightarrow$ *median*, $(c) \rightarrow$ *min*, $(d) \rightarrow$ *mean*



Preliminaries
Classification and shapes of distributions

Classification

Motivation: Another motivation: classification

- Two classes defined in terms of normal distributions (obtained from real data or directly from the parameters of the distribution $N(\mu, \Sigma)$).
- An element x in $\mathbb{R}^2$

Classification

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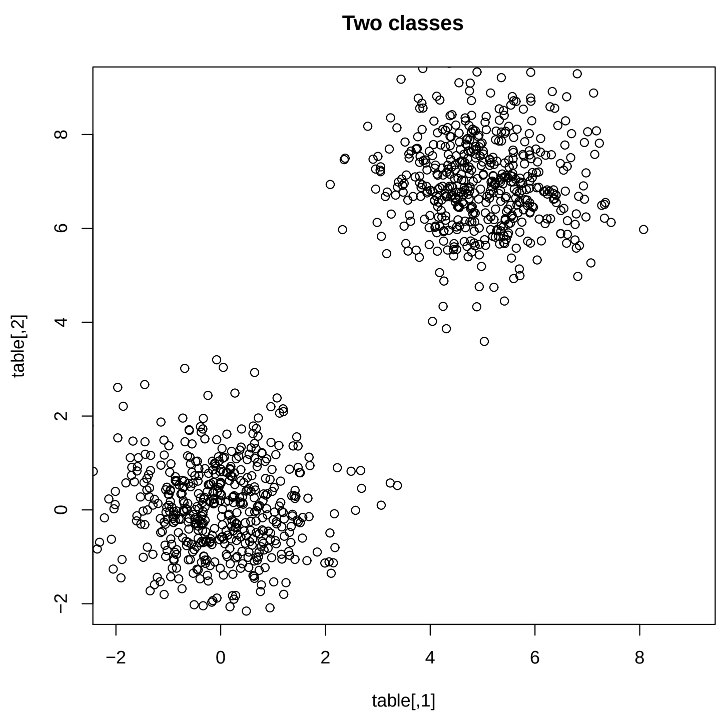
- An element x in $\mathbb{R}^2$
  $\rightarrow$ where to classify x?



Classification

Classification problems: Classification of x into Ω

- x in a n-dimensional space (*i.e.*, $x \in \mathbb{R}^n$)
- Set of k classes $\Omega = \{\omega_1, \ldots, \omega_k\}$

Formalization:

- Bayes' *maximum-a-posteriori* (MAP) classification decision rule: assigns x to the class ω_i s.t. the probability $P(\omega_i|x)$ is maximized. I.e., (Bayes condition):

$$P(\omega_i|x) = \frac{P(x|\omega_i)P(\omega_i)}{P(x)}$$

or, as $P(x)$ is constant for all classes,

$$d_i(x) = P(x|\omega_i)P(\omega_i)$$

■ $f \circ d$ results into the same classification as for d (e.g. $f = \ln$)

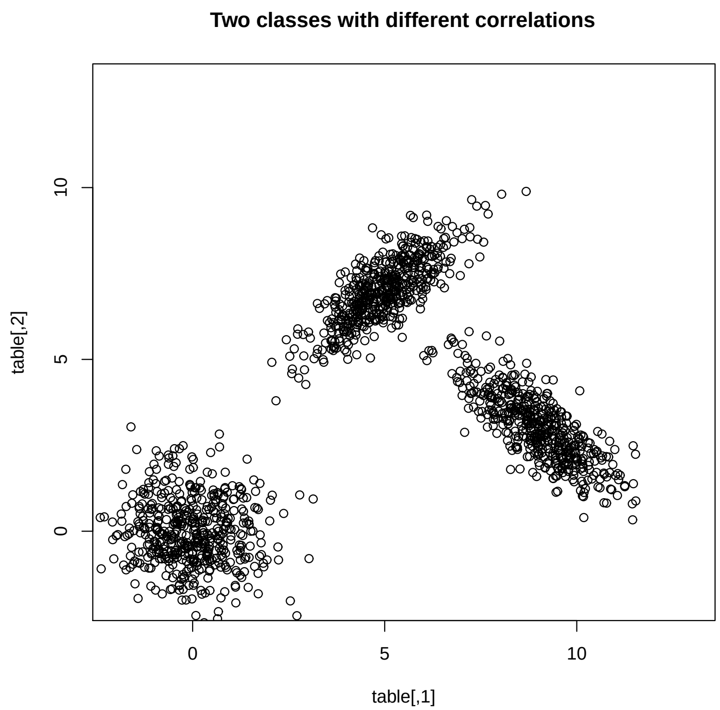
Classification

Classification problems: Classes ω_i generated from

- covariance matrices Σ_i
- means $\bar{x}_i$

$\rightarrow$ class-conditional probability-density function (Gaussian distribution)

$$P(x|\omega_i) = \frac{1}{(2\pi)^{m/2}|\Sigma_i|^{1/2}} e^{-\frac{1}{2}(x-\bar{x}_i)^T \Sigma_i^{-1} (x-\bar{x}_i)}$$



Classification

Proposition:

- Bayes' *maximum-a-posteriori* (MAP) classification decision rule, when $\Sigma_i = \Sigma_j$, and $P(\omega_i) = P(\omega_j)$, is (Mahalanobis distance)

$$d_i(x) = -(x - \bar{x}_i)^T \Sigma^{-1} (x - \bar{x}_i)$$

Classification

Proposition. Bayes' *maximum-a-posteriori* (MAP) classification

- If $\Sigma_i = \mathbb{I}$ for all i (the identity function)

$$d_i(x) = -(x - \bar{x}_i)^T(x - \bar{x}_i) = -||x - \bar{x}_i||^2$$

$\rightarrow$ Euclidean distance

- If Σ_i is diagonal (not necessarily equal to $\mathbb{I}$)

$$d_i(x) = -\sum_{j=1}^{m} (\sigma_j^2)^{-1}(x_j - \bar{x}_{ij})^2$$

$\rightarrow$ Weighted Euclidean distance

(with weights equal to the inverse of the variances: $p_j = (\sigma_j^2)^{-1}$)

Shape of distributions

The class-conditional probability-density functions established above define level curves with the shape of an ellipse
$\rightarrow$ circumference when variables are independent

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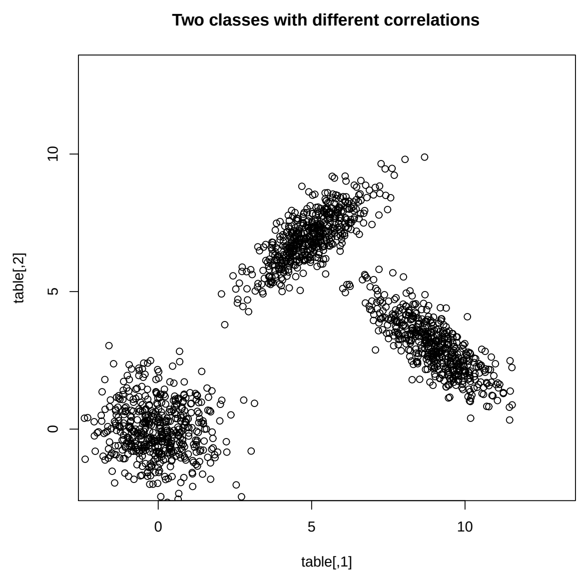

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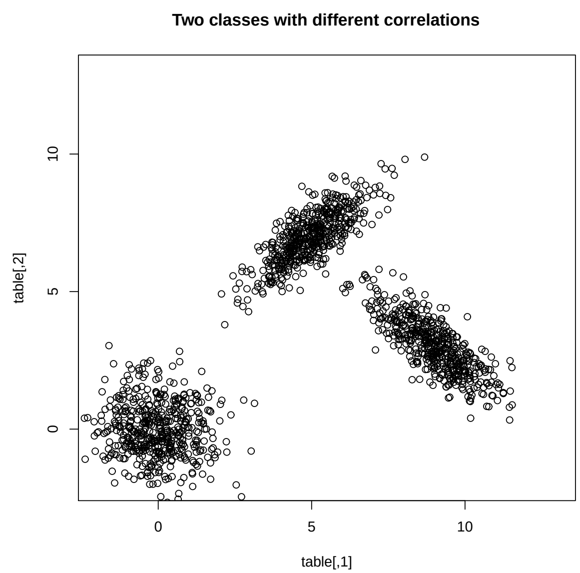

What about another shape / another distance ?

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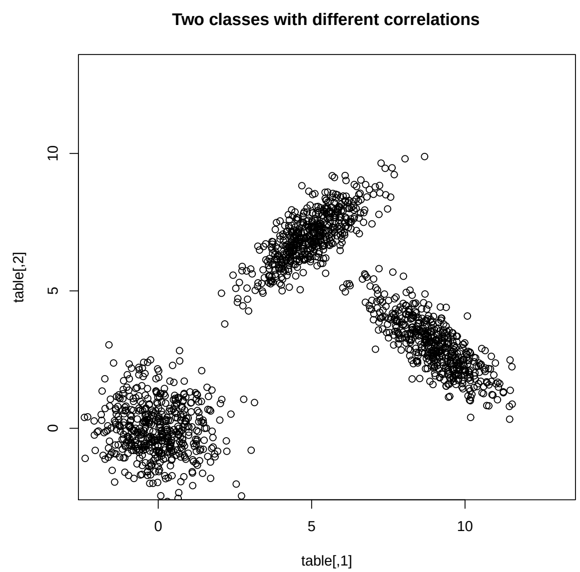

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What about using the Choquet integral here ?

Shape of distributions

Why Choquet integral?:

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  ... similar to covariances !!

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- Choquet integral integrates a function with respect to a non-additive measure
  $\rightarrow$ can it be used to compute a distance / to define a distribution ?
  $\rightarrow$ if so, what is the shape of the distribution ?

Choquet integral based distribution

Choquet integral based distribution: Definition

Definition:

- $Y = \{Y_1, \dots, Y_n\}$ random variables; $\mu : 2^Y \to [0, 1]$ a non-additive measure and $\mathbf{m}$ a vector in $\mathbb{R}^n$.
- The exponential family of Choquet integral based class-conditional probability-density functions is defined by:

$$PC_{\mathbf{m},\mu}(\mathbf{x}) = \frac{1}{K} e^{-\frac{1}{2} CI_\mu((\mathbf{x}-\mathbf{m}) \circ (\mathbf{x}-\mathbf{m}))}$$

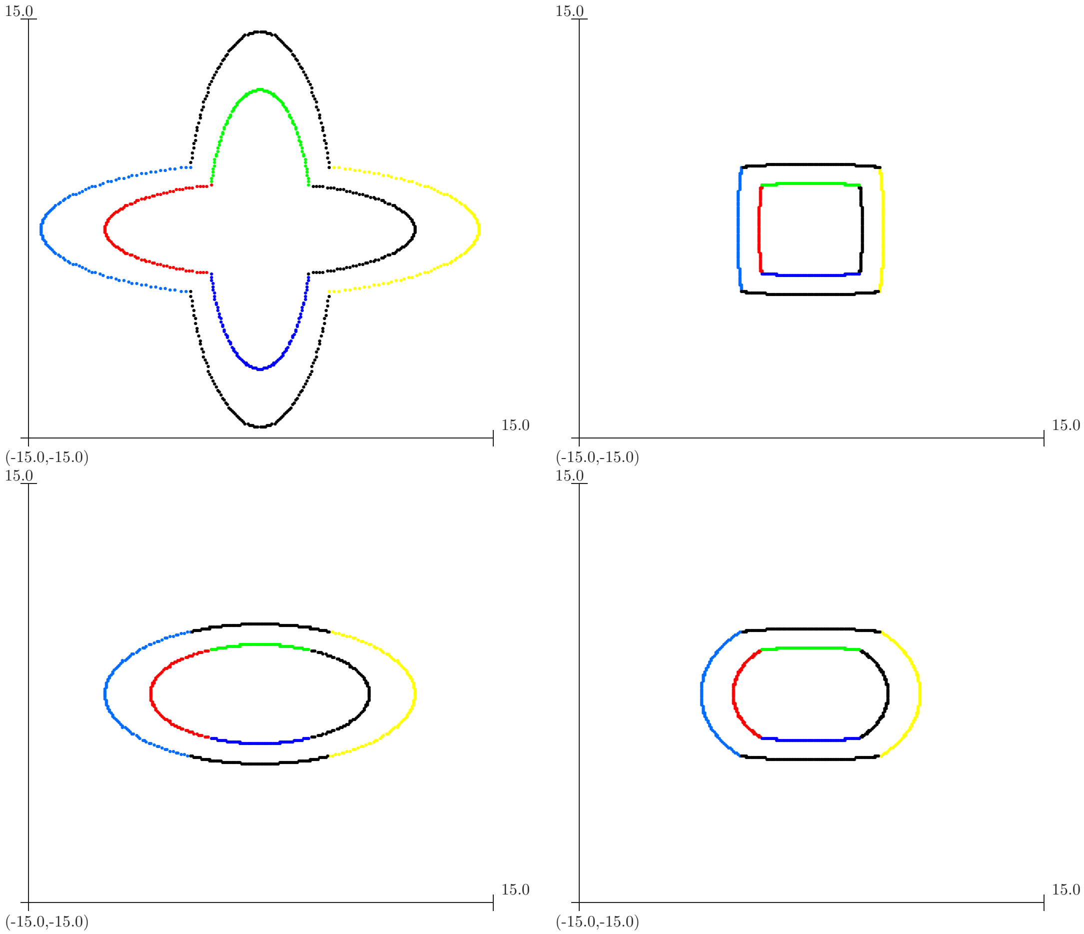
where K is a constant that is defined so that the function is a probability, and where $\mathbf{v} \circ \mathbf{w}$ denotes the Hadamard or Schur (elementwise) product of vectors $\mathbf{v}$ and $\mathbf{w}$ (i.e., $(\mathbf{v} \circ \mathbf{w}) = (v_1 w_1 \dots v_n w_n)$).

Notation:

- We denote it by $C(\mathbf{m}, \mu)$.

Choquet integral based distribution: Examples

- Shapes (level curves)



(a) $\mu_A(\{x\}) = 0.1$ and $\mu_A(\{y\}) = 0.1$, (b) $\mu_B(\{x\}) = 0.9$ and $\mu_B(\{y\}) = 0.9$, (c) $\mu_C(\{x\}) = 0.2$ and $\mu_C(\{y\}) = 0.8$, and (d) $\mu_D(\{x\}) = 0.4$ and $\mu_D(\{y\}) = 0.9$.

Choquet integral based distribution: Properties

Proposition. Distribution and distance (Choquet distance):

- If $P(w_i) = P(w_j)$ holds for all $i \neq j$, the decision rule is (max):

$$-CI_\mu((x - \bar{x}_i) \otimes (x - \bar{x}_i))$$

Proposition: Distribution/distance and level curves:

- The level curves of the Choquet integral in two variables $X = \{x, y\}$ corresponds to an ellipse when $\mu(\{x\}) = 1 - \mu(\{y\})$.
  $\rightarrow$ A natural result: we have an ellipse when $\mu(\{x\}) + \mu(\{y\}) = 1$
  $\rightarrow$ i.e., when μ is a probability.
  This follows from the fact that the Choquet integral with a measure that is a probability is equivalent to a weighted mean. Then, similar results are obtained for larger dimensions.

Choquet integral based distribution: Properties

Property:

- The family of distributions $N(\mathbf{m}, \mathbf{\Sigma})$ in $\mathbb{R}^n$ with a diagonal matrix Σ of rank n, and the family of distributions $C(\mathbf{m}, \mu)$ with an additive measure μ with all $\mu(\{x_i\}) \neq 0$ are equivalent.

($\mu(X)$ is not necessarily here 1)

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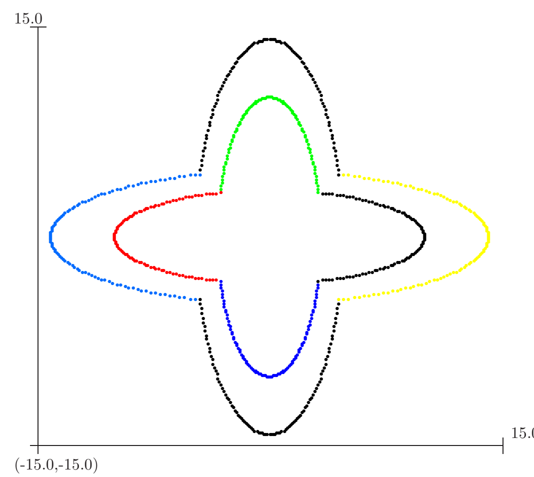
Corollary:

- The distribution $N(\mathbf{0}, \mathbb{I})$ corresponds to $C(\mathbf{0}, \mu^1)$ where μ^1 is the additive measure defined as $\mu^1(A) = |A|$ for all $A \subseteq X$.

Choquet integral based distribution: N vs. C

Properties:

- In general, the two families of distributions $N(\mathbf{m}, \mathbf{\Sigma})$ and $C(\mathbf{m}, \mu)$ are different.
- $C(\mathbf{m}, \mu)$ always symmetric w.r.t. Y_1 and Y_2 axis.



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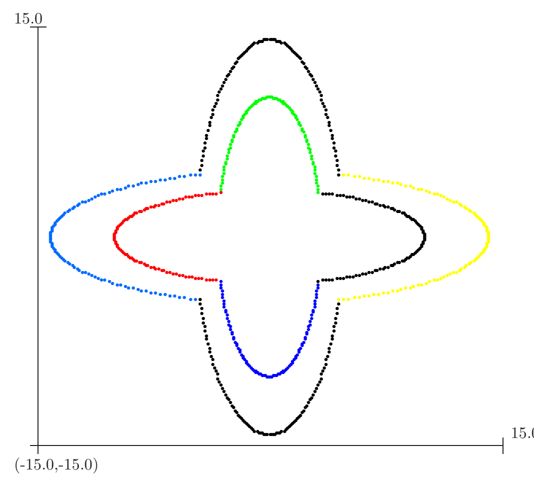
- In general, the two families of distributions $N(\mathbf{m}, \mathbf{\Sigma})$ and $C(\mathbf{m}, \mu)$ are different.
- $C(\mathbf{m}, \mu)$ always symmetric w.r.t. Y_1 and Y_2 axis.



- A generalization of both: Choquet-Mahalanobis based distribution.
  - Mahalanobis: Σ represents some interactions
  - Choquet (measure): μ represents some interactions

Choquet-Mahalanobis based distribution

Choquet integral based distribution: generalized *distance*

Definition:

- Σ be a matrix, $\Sigma^{-1} = LL^*$ be the Cholesky decomposition of its inverse.
- The Choquet-Mahalanobis *integral* is defined by

$$CMI_{\mu,\Sigma}(x, \bar{x}) = CI_{\mu}(v \otimes w) \tag{4}$$

where v and w are the vectors defined by:

$v = (x - \bar{x})^T L$ and $w = L^*(x - \bar{x})$,

where $v \otimes w$ denotes the elementwise product of vectors v and w (i.e., $(v \otimes w) = (v_1 w_1 \ldots v_n w_n)$).

Choquet integral based distribution: generalized *distance*

On the definition:

- Well defined when Σ is a covariance matrix

  When Σ^{-1} is a definite-positive matrix, the Cholesky descomposition is unique. This is the case when Σ is a covariance matrix valid for generating a probability-density function.

Choquet integral based distribution: generalized *distance*

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  When Σ^{-1} is a definite-positive matrix, the Cholesky descomposition is unique. This is the case when Σ is a covariance matrix valid for generating a probability-density function.

Proper generalization:

- Generalization of both the Mahalanobis and the Choquet integral based distance.
  - The definition with Σ equal to the identity results into the Choquet integral of $(x - \bar{x}) \otimes (x - \bar{x})$ with respect to μ.
  - The definition with μ corresponding to an additive probability $\mu(A) = 1/|A|$ results into $1/n$ of the Mahalanobis distance with respect to Σ.

Choquet integral based distribution: Definition

Definition:

- $Y = \{Y_1, \ldots, Y_n\}$ random variables, $\mu : 2^Y \to [0,1]$ a measure, $\mathbf{m}$ a vector in $\mathbb{R}^n$, and Q a positive-definite matrix.
- The exponential family of Choquet-Mahalanobis integral based class-conditional probability-density functions is defined by:

$$PCM_{\mathbf{m},\mu,\mathbf{Q}}(x) = \frac{1}{K} e^{-\frac{1}{2} CI_\mu(\mathbf{v} \circ \mathbf{w})}$$

where K is a constant that is defined so that the function is a probability, where $\mathbf{L}\mathbf{L}^T = \mathbf{Q}$ is the Cholesky decomposition of the matrix $\mathbf{Q}$, $\mathbf{v} = (\mathbf{x} - \mathbf{m})^T \mathbf{L}$, $w = \mathbf{L}^T(\mathbf{x} - \mathbf{m})$, and where $\mathbf{v} \circ \mathbf{w}$ denotes the elementwise product of vectors $\mathbf{v}$ and $\mathbf{w}$.

Notation:

- We denote it by $CMI(\mathbf{m}, \mu, \mathbf{Q})$.

Choquet integral based distribution: Properties

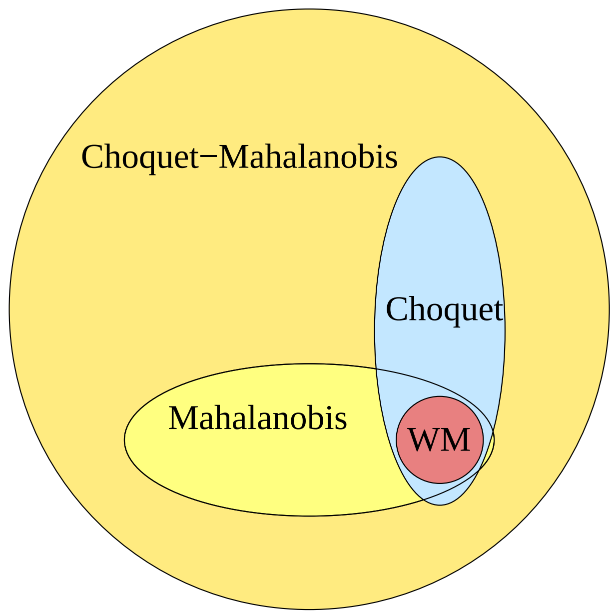
Property:

- The distribution $CMI(\mathbf{m}, \mu, \mathbf{Q})$ generalizes the multivariate normal distributions and the Choquet integral based distribution. In addition
  - A $CMI(\mathbf{m}, \mu, \mathbf{Q})$ with $\mu = \mu^1$ corresponds to multivariate normal distributions,
  - A $CMI(\mathbf{m}, \mu, \mathbf{Q})$ with $Q = \mathbb{I}$ corresponds to a $CI(\mathbf{m}, \mu)$.

Choquet integral based distribution: Properties

Graphically:

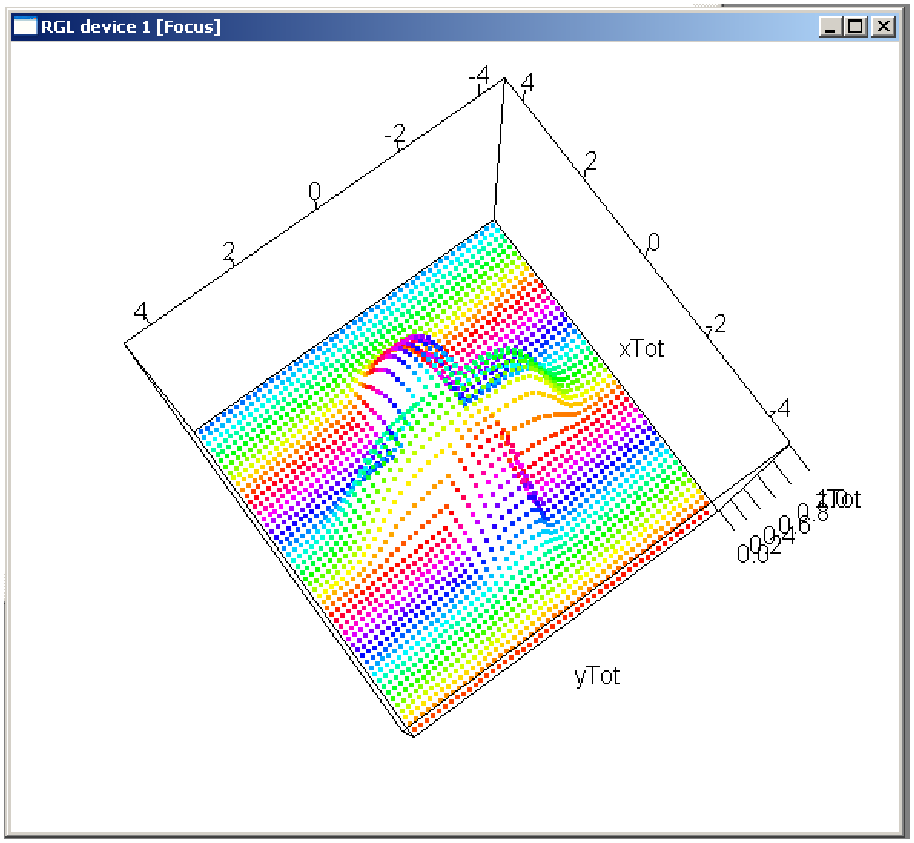
- Choquet integral (CI distribution), Mahalobis distance (multivariate normal distribution), generalization (CMI distribution)

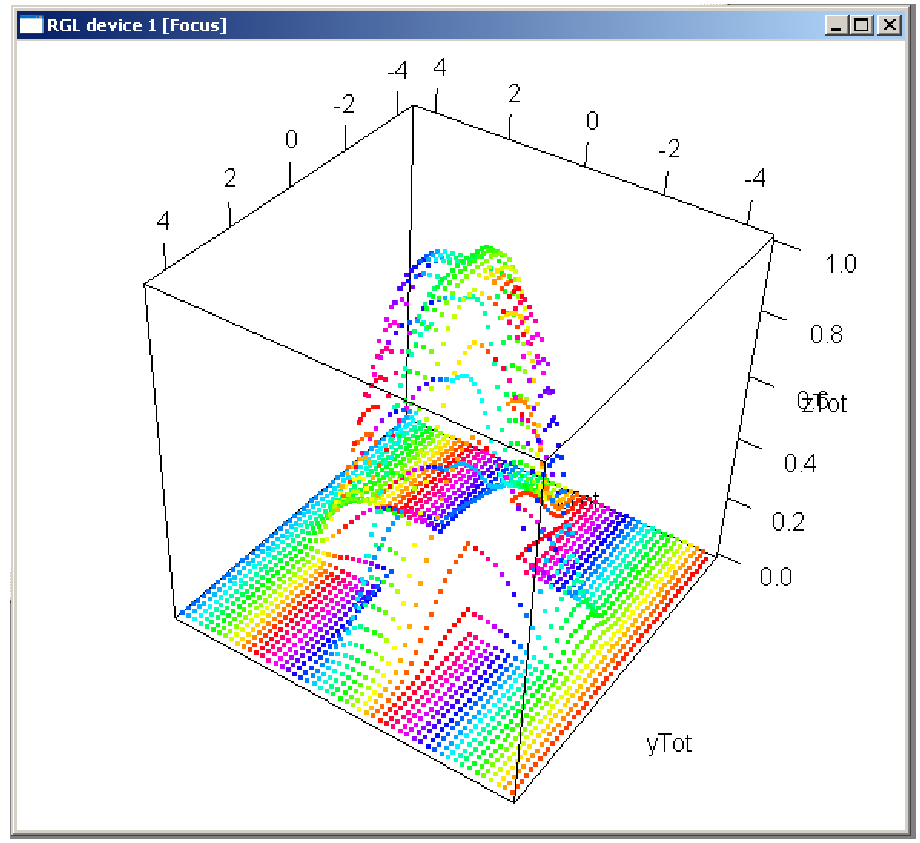


Choquet integral based distribution: Examples

1st Example: Interactions only expressed in terms of a measure.

- No correlation exists between the variables.
- CMI with $\sigma_1 = 1$, $\sigma_2 = 1$, $\rho_{12} = 0.0$, $\mu_x = 0.01$, $\mu_y = 0.01$.

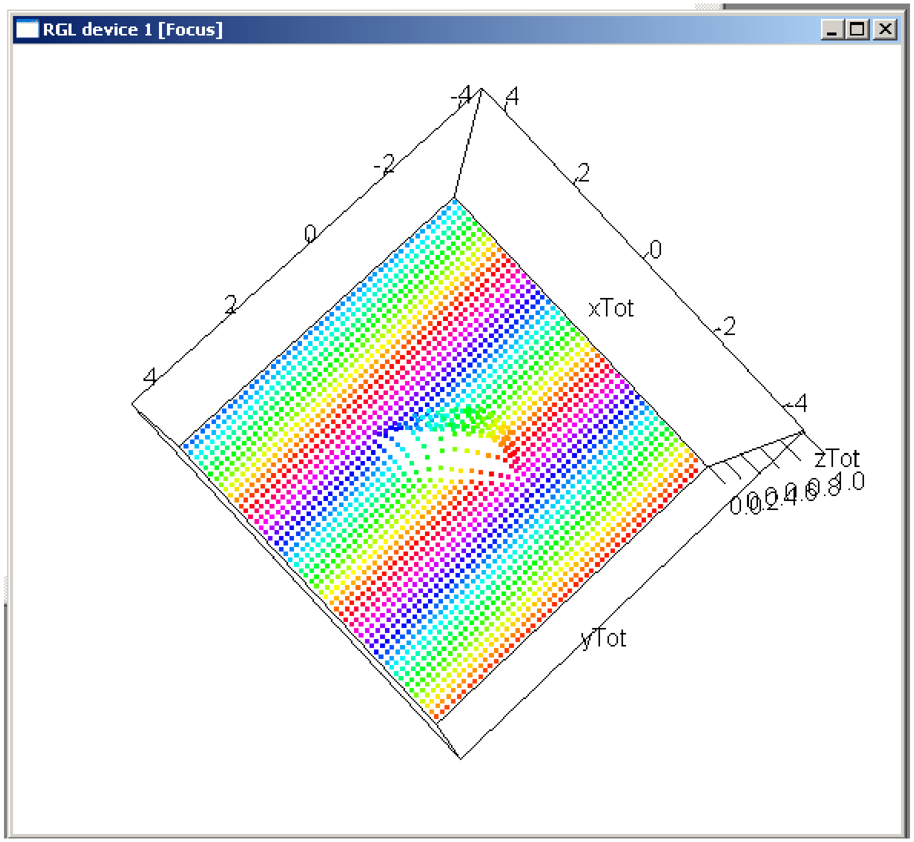


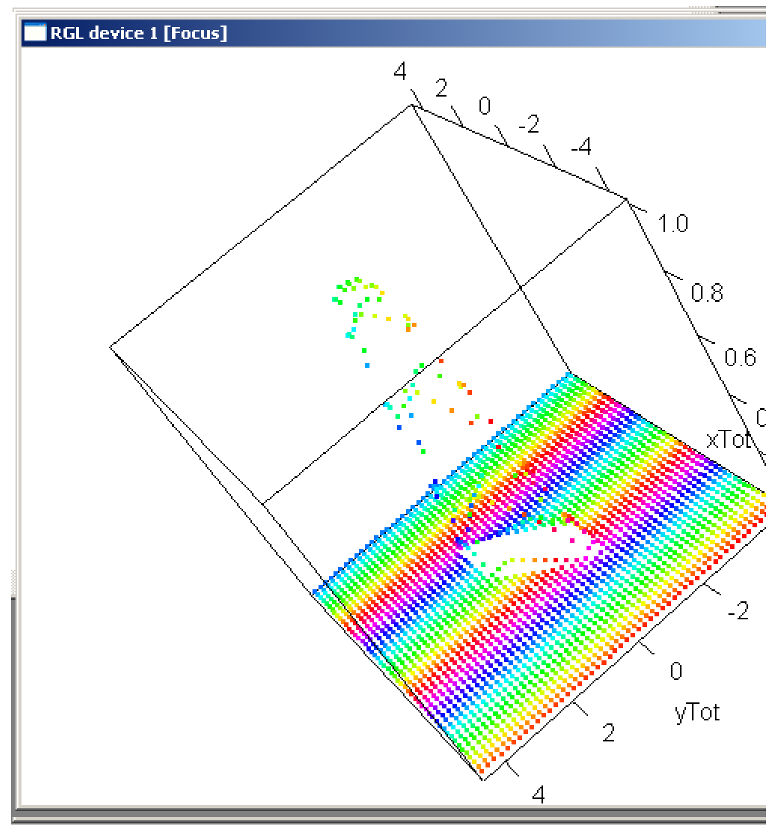


Choquet integral based distribution: Examples

2nd Example: Interactions only in terms of a covariance matrix.

- CMI with $\sigma_1 = 1$, $\sigma_2 = 1$, $\rho_{12} = 0.9$, $\mu_x = 0.10$, $\mu_y = 0.90$.

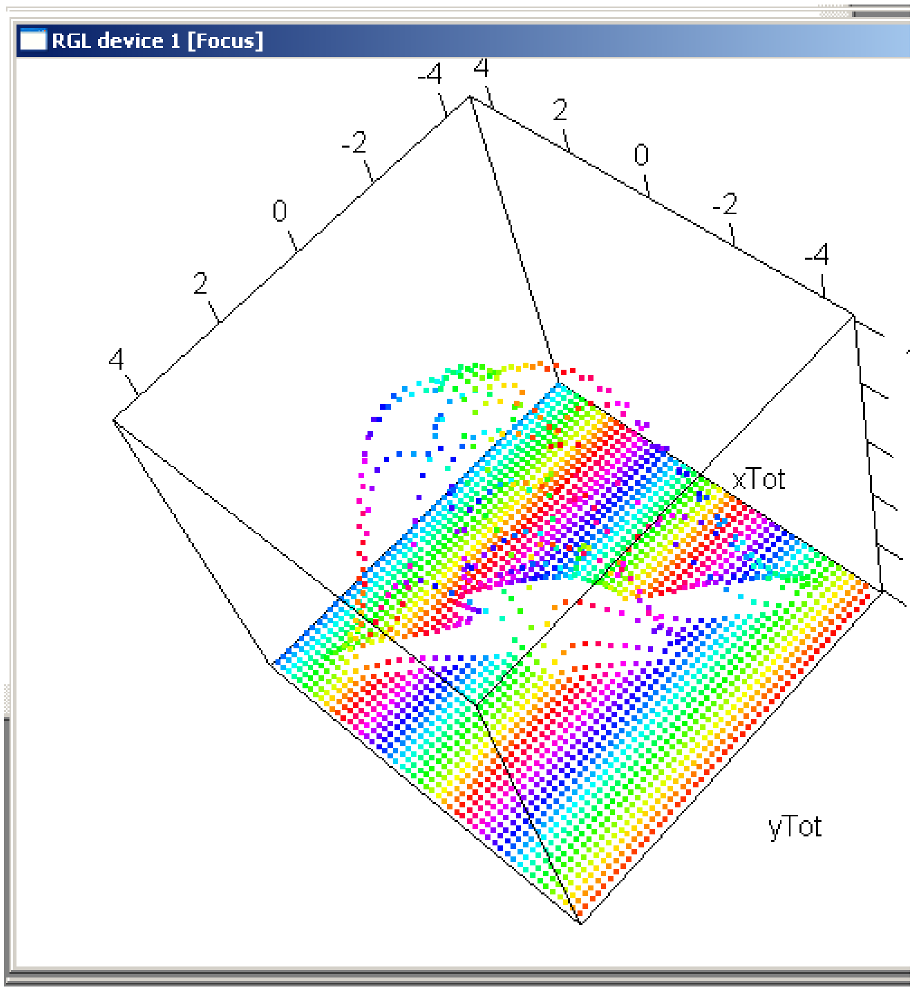


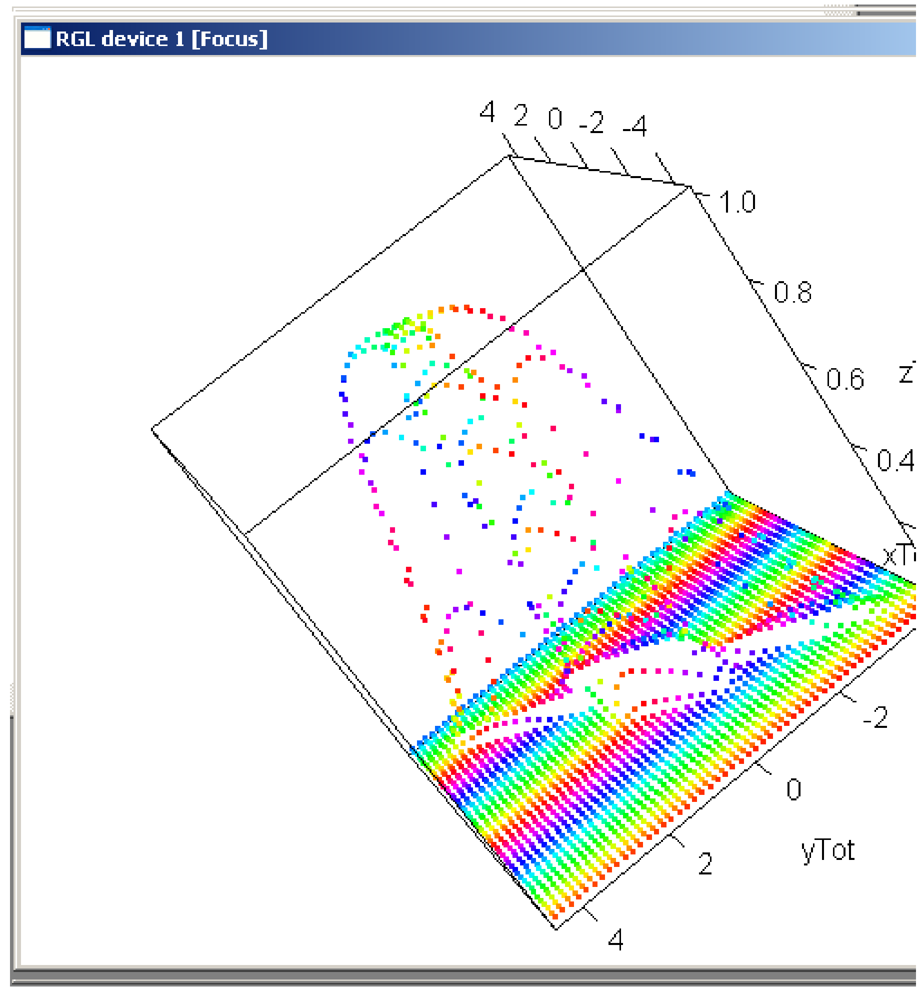


Choquet integral based distribution: Examples

3rd Example: Interactions both: covariance matrix and measure.

- CMI with $\sigma_1 = 1$, $\sigma_2 = 1$, $\rho_{12} = 0.9$, $\mu_x = 0.01$, $\mu_y = 0.01$.





Choquet integral based distribution: Properties

More properties: Data not always acc. normality assumption

- spherical, elliptical distributions
- They generalize, respectively, $N(\mathbf{0}, \mathbb{I})$ and $N(\mathbf{m}, \Sigma)$

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- Neither $CMI(\mathbf{m}, \mu, \mathbf{Q}) \subseteq / \supseteq$ spherical / elliptical distributions.

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Example:

- Non-additive μ: $CMI(\mathbf{m}, \mu, \mathbf{Q})$ not repr. spherical/elliptical

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Example:

- Non-additive μ: $CMI(\mathbf{m}, \mu, \mathbf{Q})$ not repr. spherical/elliptical
- No CMI for the following spherical distribution: Spherical distribution with density

$$f(r) = (1/K)e^{-\left(\frac{r-r_0}{\sigma}\right)^2},$$

where r_0 is a radius over which the density is maximum, σ is a variance, and K is the normalization constant.

Choquet integral based distribution: Properties

More properties: (symmetry)

- $P(x)$ a $C(\mathbf{m}, \mu)$ i.e., mean $\mathbf{m} = (m_1, \ldots, m_n)$ and a fuzzy measure μ. Then, for all $x \in \mathbb{R}^n$ and all $i \in \{1, \ldots, n\}$

$$P(x_1, \ldots, x_{i-1}, x_i + m_i, x_{i+1}, \ldots, x_n) = P(x_1, \ldots, x_{i-1}, -x_i + m_i, x_{i+1}, \ldots, x_n).$$

- $P(x)$ a $CMI(\mathbf{m}, \mu, \mathbf{Q})$ i.e., with mean $\mathbf{m} = (m_1, \ldots, m_n)$, a positive-definite diagonal matrix $\mathbf{Q}$, and a fuzzy measure μ. Then, for all $x \in \mathbb{R}^n$ and all $i \in \{1, \ldots, n\}$

$$P(x_1, \ldots, x_{i-1}, x_i + m_i, x_{i+1}, \ldots, x_n) = P(x_1, \ldots, x_{i-1}, -x_i + m_i, x_{i+1}, \ldots, x_n).$$

Choquet integral based distribution: Properties

More properties:

- $P(x)$ a $C(\mathbf{m}, \mu)$ i.e., with mean $\mathbf{m} = (m_1, \ldots, m_n)$. Then, for any fuzzy measure μ,
  - the mean vector $\bar{\mathbf{X}} = [E[X_1], E[X_2], \ldots, E[X_n]]$ is $\mathbf{m}$ and
  - $\Sigma = [Cov[X_i, X_j]]$ for $i = 1, \ldots, n$ and $j = 1, \ldots, n$ is zero for all $i \neq j$ and, thus, diagonal.
- $P(x)$ a $CMI(\mathbf{m}, \mu, \mathbf{Q})$ i.e., with mean $\mathbf{m} = (m_1, \ldots, m_n)$. Then, for any fuzzy measure μ and any diagonal matrix Q,
  - the mean vector $\bar{\mathbf{X}} = [E[X_1], E[X_2], \ldots, E[X_n]]$ is m and
  - $\Sigma = [Cov[X_i, X_j]]$ for $i = 1, \ldots, n$ and $j = 1, \ldots, n$ is zero for all $i \neq j$ and thus, diagonal.

Choquet integral based distribution: Properties

More properties:

- When $\mathbf{Q}$ is not diagonal, we may have

$$Cov[X_i, X_j] \neq Q(X_i, X_j).$$

Choquet integral based distribution: Properties

More properties: If this type of data distinguishable from Normal ?

Choquet integral based distribution: Properties

More properties: If this type of data distinguishable from Normal ?

Study:

- Case of $X = \{x_1, x_2\}$
- $CMI(0, \mu)$ with $\mu(\{x\}) = i/10$ and $\mu(\{y\}) = i/10$ for $i = 1, 2, \ldots, 9$

Choquet integral based distribution: Properties

More properties: If this type of data distinguishable from Normal ?

Study:

- Case of $X = \{x_1, x_2\}$
- $CMI(0, \mu)$ with $\mu(\{x\}) = i/10$ and $\mu(\{y\}) = i/10$ for $i = 1, 2, \ldots, 9$
- Test: Normality test for CI-based distribution
  - Normality of the marginals
  - Normality of the multidimensional distribution

Choquet integral based distribution: Properties

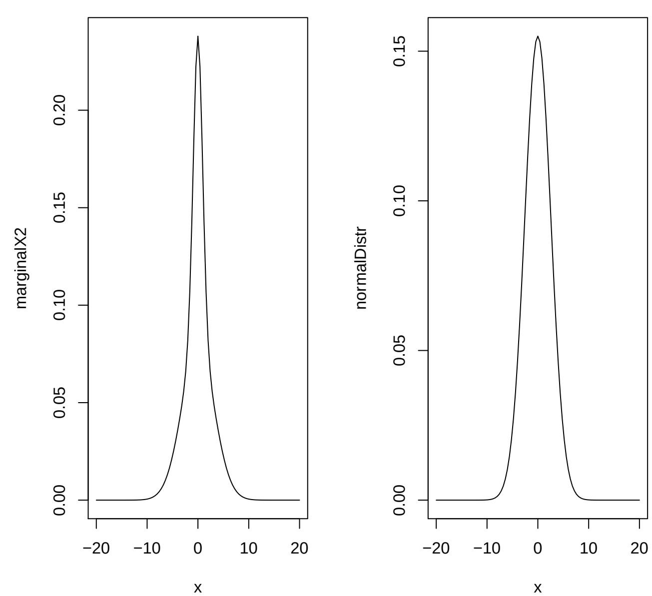
More properties: Normality test for CI-based distribution

- **Normality of the marginals:** Shapiro-Wilk test
  Marginal computed numerically `integrate`, `uniroot` function in R.
  Almost always the test is passed for samples of $n = 100$ data

Choquet integral based distribution: Properties

More properties: Normality test for CI-based distribution

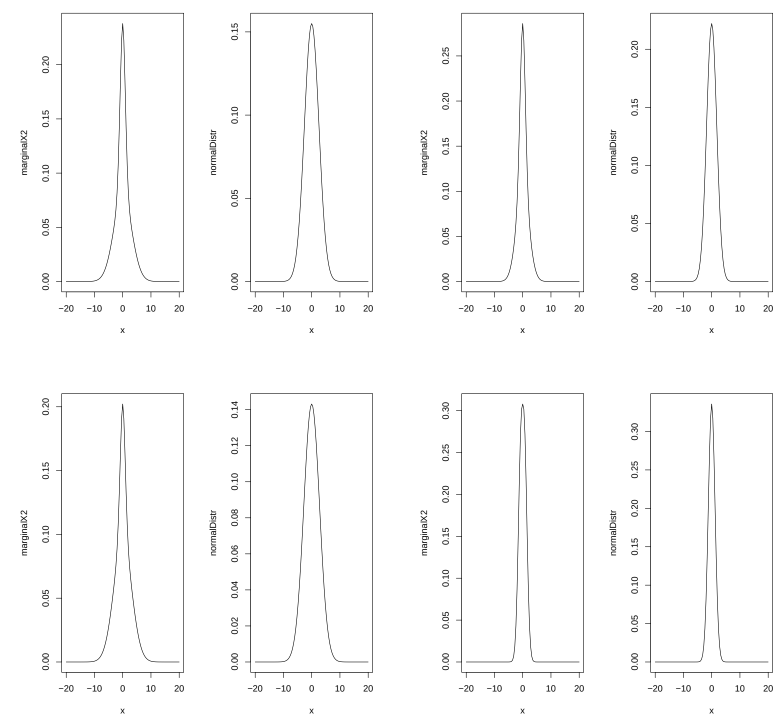
- **Normality of the marginals:** Shapiro-Wilk test
  Marginal computed numerically `integrate`, `uniroot` function in R.
  Almost always the test is passed for samples of $n = 100$ data
- Marginals (left) of the bivariate $CI(\mathbf{0}, \mu)$, and the normal distribution (right) with the same variance. $\mu(\{x_1\}) = 0.1$ and $\mu(\{x_2\}) = 0.1$



Choquet integral based distribution: Properties

More properties: Normality test for CI-based distribution

- **Normality of the marginals:** Shapiro-Wilk test
- Marginals (left) of $CI(\mathbf{0}, \mu)$, and (right) N same variance. (i) $\mu(\{x_1\}) = 0.1$ and $\mu(\{x_2\}) = 0.1$; (ii) $\mu(\{x_1\}) = 0.1$ and $\mu(\{x_2\}) = 0.2$; (iii) $\mu(\{x_1\}) = 0.2$ and $\mu(\{x_2\}) = 0.1$; (iv) $\mu(\{x_1\}) = 0.9$ and $\mu(\{x_2\}) = 0.9$



Choquet integral based distribution: Properties

More properties: Normality test for CI-based distribution

- **Normality of the distribution:**
  Mardia's test based on skewness and kurtosis
  - Skewness test is passed.
  - Almost all distributions (in $\mathbb{R}^2$) pass kurtosis test in experiments:
    - $CI(0, \mu)$ distributions with $\mu(\{x\}) = i/10$ and $\mu(\{y\}) = i/10$ for $i = 1, 2, \ldots, 9$.
    - Test only fails in
      (i) $\mu(\{x\}) = 0.1$ and $\mu(\{y\}) = 0.1$,
      (ii) $\mu(\{x\}) = 0.2$ and $\mu(\{y\}) = 0.1$.

Summary

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- Definition of distributions based on the Choquet integral
  Integral for non-additive measures
- Relationship with multivariate normal and spherical distributions

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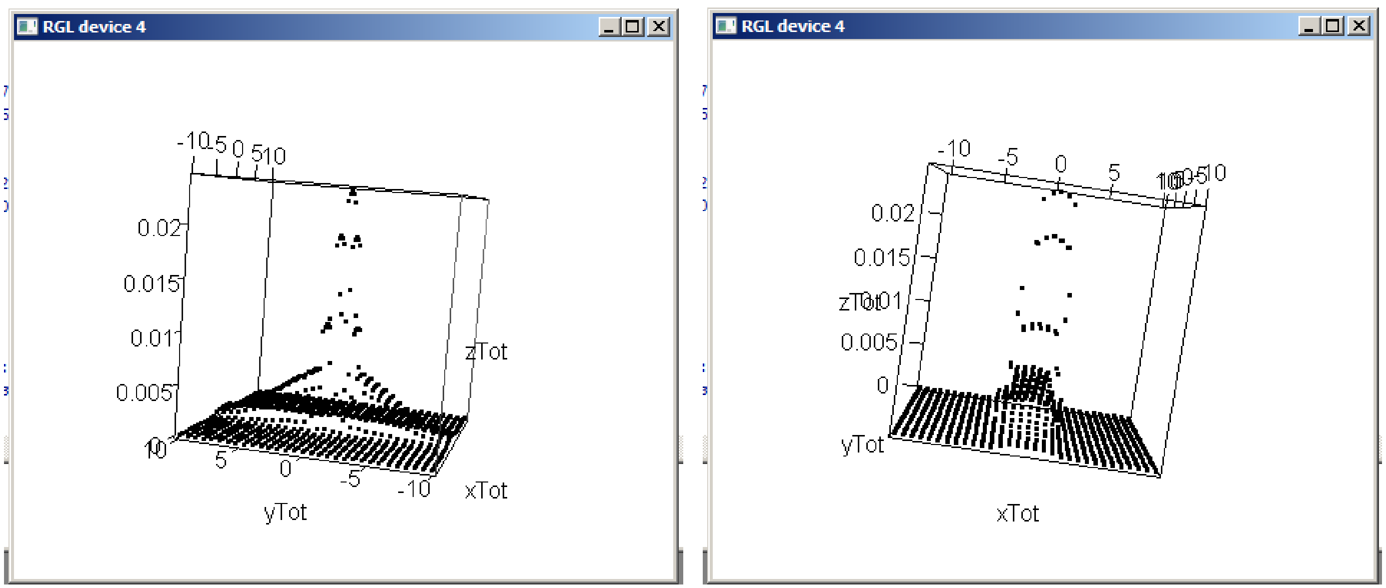
Future work:

- Study of the properties
- Parameters determination from data (μ, Q)
- Statistical tests

Summary

- Level-dependent capacity (non-additive, fuzzy measure)
  Defined by S. Greco, B. Matarazzo, S. Giove (FSS, 2011)
  - Level-dependent-based distribution (generalizes CI-based)
    $P(x) = \frac{1}{K} e^{-\frac{1}{2} CI^{G}_{\mu^{G}}((x-\bar{x})\otimes(x-\bar{x}))}$
  - Example. Two perspectives of same level dependent CI. Defined by the same fuzzy measures μ^1 and μ^2 with intervals $(0,3)$ for μ^1, and $(3,100)$ for μ^2.



$\mu^1(\{x\}) = 0.05$ and $\mu^1(\{y\}) = 0.95$, and $\mu^2(\{x\}) = 0.95$ and $\mu^2(\{y\}) = 0.05$

Thank you

References

References:

- V. Torra, Y. Narukawa, On a comparison between Mahalanobis distance and Choquet integral: the Choquet-Mahalanobis operator, Information Sciences 190 (2012) 56-63.
- V. Torra, Distributions based on the Choquet integral and non-additive measures, RIMS Kokyuroku 1906 (2014) 136-143.
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Thank you

Slides at:

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