Jaén

Funciones de agregación para la toma de decisiones multicriterio*

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*Torra, Narukawa (2007) Modeling decisions, Springer; Torra (2015) Matemáticas en las urnas, RBA

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- Toma de decisiones
  - Toma de decisiones multicriterio
- Funciones de agregación: una introducción
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Introducción

- Toma de decisiones
  - Escoger entre varias alternativas
- Un ejemplo:
  - Queremos comprar un coche y hay varios modelos
  - Alternativas: $\{Peugeot308, FordT., \ldots\}$
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Introducción

- Marco general de la toma de decisiones
  - Características del problema
    - Varias alternativas
    - $\{Peugeot308, FordT., \ldots\}$

Introducción

- Marco general de la toma de decisiones
  - Dificultades del problema
    - Criterios en contradicción
    - Incertidumbre y el riesgo
    - Adversario

Introducción

- Marco general de la toma de decisiones
  - Dificultad: Criterios en contradicción
    - No es posible encontrar una alternativa que satisfaga todos los criterios
    - Un coche barato y asequible pero no tan confortable
    - Precio vs. seguridad y confort

Introducción

- Marco general de la toma de decisiones
  - Dificultad: Incertidumbre y riesgo
    - Conocemos o no el efecto de nuestra acción
    - Cuando escogemos un coche sabemos su precio y la capacidad del maletero
    - Cuando compramos un boleto de loteria, no sabemos si ganaremos
    - Cuando el médico propone un tratamiento, no está seguro su efecto

Introducción

- Marco general de la toma de decisiones
  - Dificultad: Decisiones con adversario
    - Nuestra decisión debe confrontarse con la de e.g. oponentes
    - Los juegos con adversario:
      a nuestro movimiento le sigue el del adversario

Introducción

- Algunas notas sobre el marco general de la toma de decisiones
  - Incertidumbre vs. riesgo: conceptos diferentes

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    - Decisión bajo riesgo:
      - ∗ Cada acción conduce a varios estados con probabilidades conocidas
        - · Caso de la loteria
        - · Caso de los juegos (con dados)

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    - Decisión bajo riesgo:
      - ∗ Cada acción conduce a varios estados con probabilidades conocidas
        - · Caso de la loteria
        - · Caso de los juegos (con dados)
    - Decisión bajo incertidumbre:
      - ∗ Las probabilidades son desconocidas o no comparables
        - · Caso del médico
      - ∗ No únicamente probabilidades, información vaga o imprecisa.
        - · Un poco de fiebre: alrededor de 38?

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- Marco general de la toma de decisiones: clasificación (I)
  - Toma de decisiones con certidumbre
  - Toma de decisiones con incertidumbre y riesgo
  - Toma de decisiones con adversario

Introducción

- Marco general de la toma de decisiones: clasificación (II)
  - Toma de decisiones con certidumbre
    - Toma de decisiones:
      - Varias alternativas, cada una de ellas evaluada de acuerdo con varios criterios. Efectos de la decisión sin incertidumbre.
      - Ejemplo. Alternativas (coches) y criterios (precio, confort, etc.)

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      - Número infinito de alternativas: toma de decisión multiobjetivo

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      - · Ejemplo. Alternativas (coches) y criterios (precio, confort, etc.)
      - · Número finito de alternativas: toma de decisión multicriterio
      - · Número infinito de alternativas: toma de decisión multiobjetivo
  - MCDA: Herramientas para capturar, entender y analizar las diferencias
    (punto de vista constructivo)
  - MCDM: Herramientas para describir el proceso de decisión. Se supone que el proceso se puede formalizar.
    (punto de vista descriptivo)

Introducción

- Marco general de la toma de decisiones: clasificación (III)
  - Toma de decisiones con certidumbre
    - Multicriteria Decision Aid (MCDA):
      finito / punto de vista descriptivo / modelización
    - Multicriteria Decision Making (MCDM):
      finito / punto de vista constructivo
    - Multiobjective Decision Making (MODM):
      infinito

Introducción

- Ejemplo. Multiobjective decision making:
  número infinito de alternativas
  - Selección de las cantidades de carbón de dos tipos para la generación de electricidad (Dallenbach, 1994, p.314).

Introducción

- Ejemplo. Multiobjective decision making:
  número infinito de alternativas
  - Selección de las cantidades de carbón de dos tipos para la generación de electricidad (Dallenbach, 1994, p.314).
    - ◦ Vapor máximo (producción)? Beneficio máximo?
    - ◦ Tenemos que tener en cuenta restricciones
    - ◦ Cada carbón tiene sus inconvenientes (emisiones diferentes)
    - ◦ No pueden generarse emisiones en exceso
    - ∗ Formulación/resolución mediante optimización (e.g., Simplex)

Introducción

- Ejemplo. Multicriteria decision making:
  número finito de alternativas
  - La compra del coche
    - Alternativas: $\{Peugeot308, FordT., \ldots\}$
    - Puntos de vista/criterios: Precio, calidad, confort
      $\rightarrow$ representamos nuestras preferencias sobre las alternativas

Introducción

- Marco general de la toma de decisiones: clasificación (III)
  - Toma de decisiones con adversario
    - Juegos estáticos: los jugadores actuan a la vez
      Teoria de juegos (game theory), juegos no cooperativos, juegos cooperativos
    - Juegos dinámicos: los jugadores actuan secuencialmente
      Algoritmos de juegos (minimax, poda α-β)

MCDM:

Multicriteria decision making

MCDM

- Representación de preferencias
  - Funciones de utilidad.
    - Una función para cada criterio
    - La función se aplica a cada alternativa
    - El valor de la función es mayor, como mayor es la satisfacción del criterio

MCDM

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  - Relaciones de preferencia (comparación entre varias alterntivas)
    - Relación binaria para cada criterio
    - Cada relación nos ordena las alternativas

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    - Relación binaria para cada criterio
    - Cada relación nos ordena las alternativas
- Funciones de utilidad como descripción matemática de las relaciones de preferencia

MCDM

- Representación de preferencias
  - Funciones de utilidad.
    - Ford T: $U_{precio} = 0.2$, $U_{calidad} = 0.8$, $U_{confort} = 0.3$
    - Peugeot308: $U_{precio} = 0.7$, $U_{calidad} = 0.7$, $U_{confort} = 0.8$
  - Relaciones de preferencia (comparación entre varias alterntivas)
    - $\mathbf{R}_{precio}$: $R_{precio}(P308, FordT)$, $\neg R_{precio}(FordT, P308)$
    - $\mathbf{R}_{calidad}$: $\neg R_{calidad}(P308, FordT)$, $R_{calidad}(FordT, P308)$
    - $\mathbf{R}_{confort}$: $R_{confort}(P308, FordT)$, $\neg R_{confort}(FordT, P308)$

MCDM

- Representación de preferencias
  - Ejemplo. Relaciones de preferencia.

	Número asientos	Seguridad	Precio	Confort	Maletero
Ford T	+	++	+	++	+
Seat 600	+++	+	+++++	+	+++
Simca 1000	+++++	+++	++++	++++	++++
VW escarabajo	++++	+++++	++	+++++	+++++
Citroën Acadiane	++	++++	+++	+++	++

MCDM

- Representación de preferencias
  - Ejemplo. Funciones de utilidad.

	Número asientos	Seguridad	Precio	Confort	Maletero
Ford T	0	20	0	20	0
Seat 600	60	0	100	0	50
Simca 1000	100	30	100	50	70
VW escarabajo	80	50	30	70	100
Citroën Acadiane	20	40	60	40	0

MCDM

- Representación de preferencias: Relaciones de preferencia
  - Formalización: Conjunto de referencia X

    Propiedades (para todo x, y, z)
    - Relación binaria: I.e., subconjunto de $R \subseteq X \times X$
    - Denotamos $x \geq y$ sii $(x, y) \in R$
    - Relación total o completa: $x \geq y$ o $y \geq x$
    - Relación transitiva: $x \geq y$, $y \geq z$ entonces $x \geq z$
    - Relación reflexiva: $x \geq x$

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    * Relación transitiva: $x \geq y$, $y \geq z$ entonces $x \geq z$
    * Relación reflexiva: $x \geq x$
  - Definición: (en toma de decisiones)
    Una relación es una relación de preferencia racional si total, transitiva i reflexiva.
  - en matemáticas: preorden total

MCDM

- Representación de preferencias
  - Ejemplo. Relaciones de preferencia racional
    Satisfacen completitud, transitividad, reflexividad

	Número asientos	Seguridad	Precio	Confort	Maletero
Ford T	+	++	+	++	+
Seat 600	+++	+	+++++	+	+++
Simca 1000	+++++	+++	++++	++++	++++
VW escarabajo	++++	+++++	++	+++++	+++++
Citroën Acadiane	++	++++	+++	+++	++

MCDM

- Representación de preferencias: Funciones de utilidad
  - Formalización: Conjunto de referencia X
    - $U : X \to D$ para un cierto dominio D
  - Representación: Una utilidad u representa una preferencia $\geq$ para todo $x, y \in X$ cuando $x \geq y$ si y solo si $u(x) \geq u(y)$.

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Ejemplo. En el precio, la utilidad no representa la relación

Es cierto $u_{precio}(Simca1000) \geq u_{precio}(Seat600)$

pero es falso *Simca 1000* $\geq$ *Seat 600*

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    Es cierto $u_{precio}(Simca1000) \geq u_{precio}(Seat600)$

    pero es falso *Simca 1000* $\geq$ *Seat 600*
  - Relación: Podemos establecer una relación entre las utilidades y las relaciones de preferencia

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    Ejemplo. En el precio, la utilidad no representa la relación

    Es cierto $u_{precio}(Simca1000) \geq u_{precio}(Seat600)$

    pero es falso *Simca 1000* $\geq$ *Seat 600*
  - Relación: Podemos establecer una relación entre las utilidades y las relaciones de preferencia
    - Teorema. Dado un conjunto de alternativas, existe una función de utilidad que representa a la relación de preferencia si y sólo si la relación de preferencia es racional.

MCDM

- Representación de preferencias: Funciones de utilidad
  - Ejemplo: definición para precio
    - Presupuesto maximo de 10000 euros.
    - Menor que 1000 es perfecto.
    - Funcion lineal entre 1000 y 10000

$$u_p(x) = \begin{cases} 100 & if \quad x \leq 1000 \\ (10000 - x)/90 & if \quad x \in (1000, 10000) \\ 0 & if \quad x \geq 10000 \end{cases}$$

MCDM

- Representación de preferencias: Funciones de utilidad
  - Ejemplo: definición para capacidad del maletero
    No siempre hay una relación monótona
    entre los valores de un criterio y la utilidad
    - El maletero óptimo es de 1 m^3.
    - Ni demasiado pequeño, ni demasiado grande

$$u_m(x) = \begin{cases} 0 & \textit{if} \quad x \leq 0.8 \\ 100 - 500|x-1| & \textit{if} \quad x \in (0.8, 1.2) \\ 0 & \textit{if} \quad x \geq 1.2 \end{cases}$$

MCDM

- Decisión
  - Modelización del problema: representación de los criterios
  - Agregación
  - Selección de las alternativas

MCDM

- Agregación, según la representación de las preferencias
  - Funciones de utilidad
    - ○ Ford T: $U_{precio} = 0.2$, $U_{calidad} = 0.8$, $U_{confort} = 0.3$
    - ∗ Dadas unas utilidades, tenemos que agregarlas
  - Relaciones de preferencia (comparación entre varias alterntivas)
    - ○ $\mathbf{R}_{precio}$: $R_{precio}(P308, FordT)$, $\neg R_{precio}(FordT, P308)$
    - ○ $\mathbf{R}_{calidad}$: $\neg R_{calidad}(P308, FordT)$, $R_{calidad}(FordT, P308)$
    - ∗ Dadas unas relaciones de preferencia, tenemos que agregarlas

MCDM

- Decisión con relaciones de preferencia
  Modelización, agregación, selección

	Número asientos	Seguridad	Precio	Confort	Maletero	Preferencia agregada
Ford T	+	++	+	++	+	+
Seat 600	+++	+	+++++	+	+++	++
Simca 1000	+++++	+++	++++	++++	++++	++++
VW esc.	++++	+++++	++	+++++	+++++	+++++
Citr. Acadiane	++	++++	+++	+++	++	+++

MCDM

- Decisión con funciones de utilidad
  Modelización, agregación = AM, selección

	Número asientos	Seguridad	Precio	Confort	Maletero	Preferencia agregada
Ford T	0	20	0	20	0	8
Seat 600	60	0	100	0	50	42
Simca 1000	100	30	100	50	70	70
VW	80	50	30	70	100	66
Citr. Acadiane	20	40	60	40	0	32

Aggregation functions: an introduction

Aggregation functions

- Aggregation and information fusion
  - In our case, how to combine information about criteria
- In general,
  - it is a broad area, with different types of applications

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  - In our case, how to combine information about criteria
- In general,
  - it is a broad area, with different types of applications
- Examples of aggregation functions:
  - $\sum_{i=1}^{N} a_i / N$ (AM arithmetic mean)
  - $\sum_{i=1}^{N} p_i \cdot a_i$ (WM weighted mean)

Aggregation functions

- Aggregation and information fusion
  - In our case, how to combine information about criteria
- In general,
  - it is a broad area, with different types of applications
- Examples of aggregation functions:
  - $\sum_{i=1}^{N} a_i/N$ (AM arithmetic mean)
  - $\sum_{i=1}^{N} p_i \cdot a_i$ (WM weighted mean)
- Different functions, lead to different results
  - In our case, different orderings, different selections!

Aggregation functions

- Goal of aggregation functions (in general, not restricted to MCDM):
  - To produce a specific datum, and exhaustive, on an entity
  - Datum produced from information supplied by different information sources (or the same source over time)
  - Techniques to reduce noise, increase precision, summarize information, extract information, make decisions, etc.

Aggregation functions

- Information fusion studies ...

  ... all aspects related to combining information:

- Goals of data aggregation (*goals of the area*):

Aggregation functions

- Information fusion studies . . .

  . . . all aspects related to combining information:

- Goals of data aggregation (*goals of the area*):
  - Formalization of the aggregation process
    - Definition of new functions
    - Selection of functions

      (methods to decide which is the most appropriate function in a given context)
    - Parameter determination

Aggregation functions

- Information fusion studies ...

  ... all aspects related to combining information:

- Goals of data aggregation (*goals of the area*):
  - Formalization of the aggregation process
    - Definition of new functions
    - Selection of functions

      (methods to decide which is the most appropriate function in a given context)
    - Parameter determination
  - Study of existing methods:
    - Caracterization of functions
    - Determination of the modeling capabilities of the functions
    - Relation between operators and parameters

      (how parameters influence the result: can be achieve dictatorship?, sensitivity to data $\rightarrow$ index).

Aggregation functions

- Terms:
  - Information integration
  - Information fusion: concrete functions / techniques

    concrete process to combine several data into a single datum.
  - Aggregation functions: $\mathbb{C} : D^N \to D$ ($\mathbb{C}$ from $\mathbb{C}$onsensus)

    $\to$ i $\mathbb{C}$ with parameters (background knowledge): $\mathbb{C}_P$

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  - Unanimity and idempotency: $\mathbb{C}(a, \ldots, a) = a$ for all a
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  - Monotonicity: $\mathbb{C}(a_1, \ldots, a_N) \geq \mathbb{C}(a'_1, \ldots, a'_N)$, if $a_i \geq a'_i$
  - Symmetry: For all permutation π over $\{1, \ldots, N\}$
    $\mathbb{C}(a_1, \ldots, a_N) = \mathbb{C}(a_{\pi(1)}, \ldots, a_{\pi(N)})$

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  - Monotonicity: $\mathbb{C}(a_1, \ldots, a_N) \geq \mathbb{C}(a'_1, \ldots, a'_N)$, if $a_i \geq a'_i$
  - Symmetry: For all permutation π over $\{1, \ldots, N\}$
    $\mathbb{C}(a_1, \ldots, a_N) = \mathbb{C}(a_{\pi(1)}, \ldots, a_{\pi(N)})$
  - Unanimity + monotonicity $\to$ internality:
    $\min_i a_i \leq \mathbb{C}(a_1, \ldots, a_N) \leq \max_i a_i$

Aggregation functions

Definition of aggregation functions:

- Definition from properties

  properties $\longrightarrow$ function

- Heuristic definition

  properties $\longleftarrow$ function

- Definition from examples

  examples $\longrightarrow$ function

Aggregation functions

- Definition from properties

$$\boxed{\text{properties}} \longrightarrow \boxed{\text{function}}$$

Aggregation functions

- Definition from properties

$$\boxed{\text{properties}} \longrightarrow \boxed{\text{function}}$$

- Some ways

  a) Using functional equations

Aggregation functions

- Definition from properties

$$\boxed{\text{properties}} \longrightarrow \boxed{\text{function}}$$

- Some ways

a) Using functional equations

b) Aggregation of a_1, a_2, ..., $a_N \in D$, as the datum c which is at a minimum distance from a_i:

$$\mathbb{C}(a_1, a_2, \ldots, a_N) = \arg\min_c \{\sum_{a_i} d(c, a_i)\},$$

d is a distance over D.

Aggregation functions

- Example (case (a)): Functional equations
  - Cauchy equation

$$\phi(x + y) = \phi(x) + \phi(y)$$

  - find ϕ !

Aggregation functions

- Example (case (a)): Functional equations
  - Cauchy equation

$$\phi(x + y) = \phi(x) + \phi(y)$$

  - find ϕ !
  - $\phi(x) = \alpha x$ for an arbitrary value for α

Aggregation functions

- Example (case (a)): Functional equations
  - distribute s euros among m projects according to the opinion of N experts

	Proj 1	Proj 2	$\cdots$	Proj j	$\cdots$	Proj m
E_1	x_1^1	x_2^1	$\cdots$	x_j^1	$\cdots$	x_m^1
E_2	x_1^2	x_2^2	$\cdots$	x_j^2	$\cdots$	x_m^2
	$\vdots$	$\vdots$		$\vdots$		$\vdots$
E_i	x_1^i	x_2^i	$\cdots$	x_j^i	$\cdots$	x_m^i
	$\vdots$	$\vdots$		$\vdots$		$\vdots$
E_N	x_1^N	x_2^N	$\cdots$	x_j^N	$\cdots$	x_m^N
DM	$f_1(\mathbf{x_1})$	$f_2(\mathbf{x_2})$	$\cdots$	$f_j(\mathbf{x_j})$	$\cdots$	$f_m(\mathbf{x_m})$

Aggregation functions

- The general solution of the system (Proposition 3.11) for a given $m > 2$

$$f_j : [0, s]^N \to \mathbb{R}^+ \text{ for } j = \{1, \cdots, m\} \quad (1)$$

$$\sum_{j=1}^{m} \mathbf{x}_j = \mathbf{s} \text{ implies that } \sum_{j=1}^{m} f_j(\mathbf{x}_j) = s \quad (2)$$

$$f_j(\mathbf{0}) = 0 \text{ for } j = 1, \cdots, m \quad (3)$$

is given by

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$$f_j(\mathbf{0}) = 0 \text{ for } j = 1, \cdots, m \quad (3)$$

is given by

$$f_1(\mathbf{x}) = f_2(\mathbf{x}) = \cdots = f_m(\mathbf{x}) = f((x_1, x_2, \ldots, x_N)) = \sum_{i=1}^{N} \alpha_i x_i, \quad (4)$$

where $\alpha_1, \cdots, \alpha_N$ are nonnegative constants satisfying $\sum_{i=1}^{N} \alpha_i = 1$, but are otherwise arbitrary.

Aggregation functions

- Example (case (b)): Consider the following expression

$$\mathbb{C}(a_1, a_2, \ldots, a_N) = \arg\min_c \{\sum_{a_i} d(c, a_i)\},$$

where a_i are numbers from $\mathbb{R}$ and d is a distance on D. Then,

Aggregation functions

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where a_i are numbers from $\mathbb{R}$ and d is a distance on D. Then,

1. When $d(a,b) = (a-b)^2$, $\mathbb{C}$ is the arithmetic mean
   I.e., $\mathbb{C}(a_1, a_2, \ldots, a_N) = \sum_{i=1}^{N} a_i/N$.
2. When $d(a,b) = |a-b|$, $\mathbb{C}$ is the median
   I.e., the median of $a_1, a_2, \ldots, a_N$ is the element which occupies the central position when we order a_i.
3. When $d(a,b) = 1$ iff $a = b$, $\mathbb{C}$ is the plurality rule (mode or voting).
   I.e., $\mathbb{C}(a_1, a_2, \ldots, a_N)$ selects the element of $\mathbb{R}$ with a largest frequency among elements in $(a_1, a_2, \ldots, a_N)$.

Aggregation for (numerical) utility functions

Aggregation for (numerical) utility functions

- Decisión con funciones de utilidad
  Modelización, agregación $= \mathbb{C}$, selección

	Seats	Security	Price	Comfort	trunk	$\mathbb{C} = AM$
Ford T	0	20	0	20	0	8
Seat 600	60	0	100	0	50	42
Simca 1000	100	30	100	50	70	70
VW	80	50	30	70	100	66
Citr. Acadiane	20	40	60	40	0	32

Aggregation for (numerical) utility functions

- MCDM: Aggregation to deal with contradictory criteria

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- But there are occasions in which ordering is clear

$$\text{when } a_i \leq b_i \text{ it is clear that } a \leq b$$

E.g.,

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Simca 1000	100	30	100	50	70	70

- Pareto dominance: Given two vectors $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$, we say that b dominates a when $a_i \leq b_i$ for all i and there is at least one k such that $a_k < b_k$.

Aggregation for (numerical) utility functions

- Pareto set, Pareto frontier, or non dominance set:

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Simca 1000	100	30	100	50	70	70
VW	80	50	30	70	100	66
Citr. Acadiane	20	40	60	40	0	32

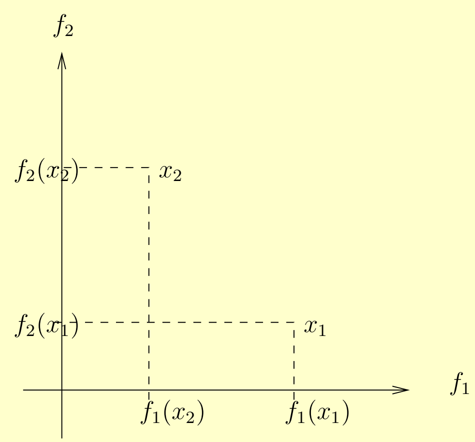
- Each one wins at least in one criteria

Aggregation for (numerical) utility functions

- Pareto set, Pareto frontier, or non dominance set:
  Given a set of alternatives U represented by vectors $u = (u_1, \ldots, u_n)$, the Pareto frontier is the set $u \in U$ such that there is no other $v \in U$ such that v dominates u.

$$PF = \{u | \text{there is no } v \text{ s.t. } v \text{ dominates } u\}$$

- Pareto optimal: an element u of the Pareto set



Aggregation for (numerical) utility functions

- MCDM: we aggregate utility, and order according to utility
- The function of aggregation functions
  - Different aggregations lead to different orders
  - Aggregation establishes which points are *equivalent*
  - Different aggregations, establish different curves of points (level curves)

Criteria
Satisfaction on:

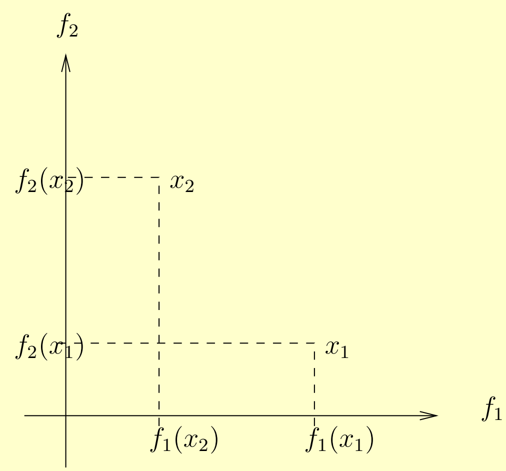
alt	Price	Quality	Comfort
FordT	0.2	0.8	0.3
206	0.7	0.7	0.8
...	...		

→

alt	Consensus
FordT	0.35
206	0.72
...	...

→

alt	Ranking
206	0.72
FordT	0.35
...	...



Aggregation for (numerical) utility functions

- Why alternatives de to the arithmetic mean?
  - Not all criteria are equally important (security and comfort)
  - There are mandatory requirements (price below a threshold)
  - Compensation among criteria
  - Interactions among criteria

Aggregation:

from the weighted mean to fuzzy integrals

Aggregation:

from the weighted mean to fuzzy integrals

An example

Aggregation: example

Example. A and B teaching a tutorial+training course w/ constraints

- The total number of sessions is six.
- Professor A will give the tutorial, which should consist of about three sessions; three is the optimal number of sessions; a difference in the number of sessions greater than two is unacceptable.
- Professor B will give the training part, consisting of about two sessions.
- Both professors should give more or less the same number of sessions. A difference of one or two is half acceptable; a difference of three is unacceptable.

Aggregation: example

Example. Formalization

- Variables
  - x_A: Number of sessions taught by Professor A
  - x_B: Number of sessions taught by Professor B
- Constraints
  - the constraints are translated into
    - C_1: $x_A + x_B$ should be about 6
    - C_2: x_A should be about 3
    - C_3: x_B should be about 2
    - C_4: $|x_A - x_B|$ should be about 0
  - using fuzzy sets, the constraints are described ...

Aggregation: example

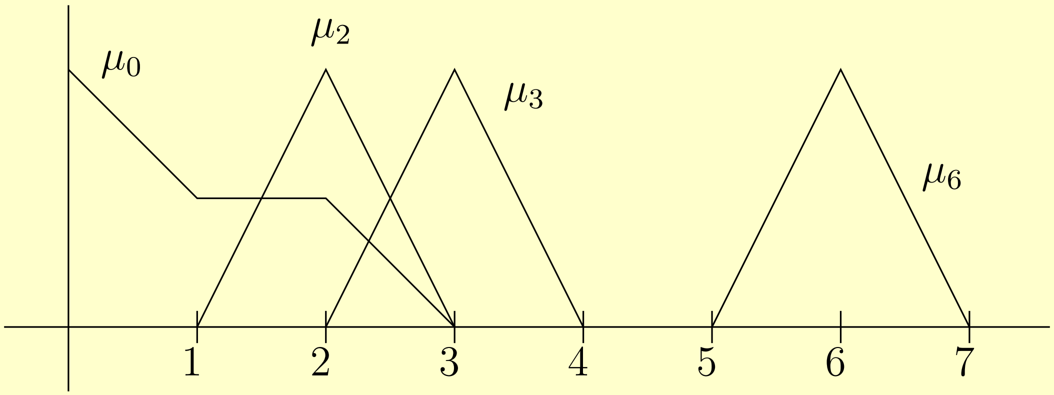
Example. Formalization

- Constraints
  - if fuzzy set μ_6 expresses "about 6," then,
    we evaluate "$x_A + x_B$ should be about 6" by $\mu_6(x_A + x_B)$.
    $\rightarrow$ given μ_6, μ_3, μ_2, μ_0,
  - Then, given a solution pair (x_A, x_B), the degrees of satisfaction:
    * $\mu_6(x_A + x_B)$
    * $\mu_3(x_A)$
    * $\mu_2(x_B)$
    * $\mu_0(|x_A - x_B|)$

Aggregation: example

Example. Formalization

- Membership functions for constraints



Aggregation: example

Example. Application

alternative	Satisfaction degrees	Satisfaction degrees			
(x_A, x_B)	$(\mu_6(x_A + x_B),\ \mu_3(x_A),\ \mu_2(x_B),\ \mu_0(\lvert x_A - x_B \rvert))$	C_1	C_2	C_3	C_4
$(2, 2)$	$(\mu_6(4),\ \mu_3(2),\ \mu_2(2),\ \mu_0(0))$	0	0.5	1	1
$(2, 3)$	$(\mu_6(5),\ \mu_3(2),\ \mu_2(3),\ \mu_0(1))$	0.5	0.5	0.5	0.5
$(2, 4)$	$(\mu_6(6),\ \mu_3(2),\ \mu_2(4),\ \mu_0(2))$	1	0.5	0	0.5
$(3.5, 2.5)$	$(\mu_6(6),\ \mu_3(3.5),\ \mu_2(2.5),\ \mu_0(1))$	1	0.5	0.5	0.5
$(3, 2)$	$(\mu_6(5),\ \mu_3(3),\ \mu_2(2),\ \mu_0(1))$	0.5	1	1	0.5
$(3, 3)$	$(\mu_6(6),\ \mu_3(3),\ \mu_2(3),\ \mu_0(0))$	1	1	0.5	1

Aggregation:

from the weighted mean to fuzzy integrals

WM, OWA, and WOWA operators

Aggregation: WM, OWA, and WOWA operators

- Operators
  - Weighting vector (dimension N): $v = (v_1 ... v_N)$ iff $v_i \in [0,1]$ and $\sum_i v_i = 1$
  - Arithmetic mean (AM :$\mathbb{R}^N \to \mathbb{R}$): $AM(a_1, ..., a_N) = (1/N) \sum_{i=1}^N a_i$
  - Weighted mean (WM: $\mathbb{R}^N \to \mathbb{R}$): $WM_{\mathbf{p}}(a_1, ..., a_N) = \sum_{i=1}^N p_i a_i$ ($\mathbf{p}$ a weighting vector of dimension N)
  - Ordered Weighting Averaging operator (OWA: $\mathbb{R}^N \to \mathbb{R}$):

$$OWA_{\mathbf{w}}(a_1, ..., a_N) = \sum_{i=1}^{N} w_i a_{\sigma(i)},$$

    where $\{\sigma(1), ..., \sigma(N)\}$ is a permutation of $\{1, ..., N\}$ s. t. $a_{\sigma(i-1)} \geq a_{\sigma(i)}$, and $\mathbf{w}$ a weighting vector.

Aggregation: WM, OWA, and WOWA operators

Example. Application

- Let us consider the following situation:
  - Professor A is more important than Professor B
  - The number of sessions equal to six is the most important constraint (not a *crisp* requirement)
  - The difference in the number of sessions taught by the two professors is the least important constraint

WM with $\mathbf{p} = (p_1, p_2, p_3, p_4) = (0.5, 0.3, 0.15, 0.05)$.

Aggregation: WM, OWA, and WOWA operators

Example. Application

- WM with $\mathbf{p} = (p_1, p_2, p_3, p_4) = (0.5, 0.3, 0.15, 0.05)$.

alternative	Aggregation of the Satisfaction degrees	WM
(x_A, x_B)	$WM_{\mathbf{p}}(C_1, C_2, C_3, C_4)$	
$(2, 2)$	$WM_{\mathbf{p}}(0, 0.5, 1, 1)$	0.35
$(2, 3)$	$WM_{\mathbf{p}}(0.5, 0.5, 0.5, 0.5)$	0.5
$(2, 4)$	$WM_{\mathbf{p}}(1, 0.5, 0, 0.5)$	0.675
$(3.5, 2.5)$	$WM_{\mathbf{p}}(1, 0.5, 0.5, 0.5)$	0.75
$(3, 2)$	$WM_{\mathbf{p}}(0.5, 1, 1, 0.5)$	0.725
$(3, 3)$	$WM_{\mathbf{p}}(1, 1, 0.5, 1)$	0.925

Aggregation: WM, OWA, and WOWA operators

Example. Application

- Compensation: how many values can have a bad evaluation
- One bad value does not matter: OWA with $\mathbf{w} = (1/3, 1/3, 1/3, 0)$ (lowest value discarded)

alternative	Aggregation of the Satisfaction degrees	OWA
(x_A, x_B)	$OWA_{\mathbf{w}}(C_1, C_2, C_3, C_4)$	
$(2, 2)$	$OWA_{\mathbf{w}}(0, 0.5, 1, 1)$	0.8333
$(2, 3)$	$OWA_{\mathbf{w}}(0.5, 0.5, 0.5, 0.5)$	0.5
$(2, 4)$	$OWA_{\mathbf{w}}(1, 0.5, 0, 0.5)$	0.6666
$(3.5, 2.5)$	$OWA_{\mathbf{w}}(1, 0.5, 0.5, 0.5)$	0.6666
$(3, 2)$	$OWA_{\mathbf{w}}(0.5, 1, 1, 0.5)$	0.8333
$(3, 3)$	$OWA_{\mathbf{w}}(1, 1, 0.5, 1)$	1.0

Aggregation: WM, OWA, and WOWA operators

- Weighted Ordered Weighted Averaging WOWA operator

  (WOWA :$\mathbb{R}^N \to \mathbb{R}$):

$$WOWA_{\mathbf{p},\mathbf{w}}(a_1, ..., a_N) = \sum_{i=1}^{N} \omega_i a_{\sigma(i)}$$

where

$$\omega_i = w^*(\sum_{j \leq i} p_{\sigma(j)}) - w^*(\sum_{j < i} p_{\sigma(j)}),$$

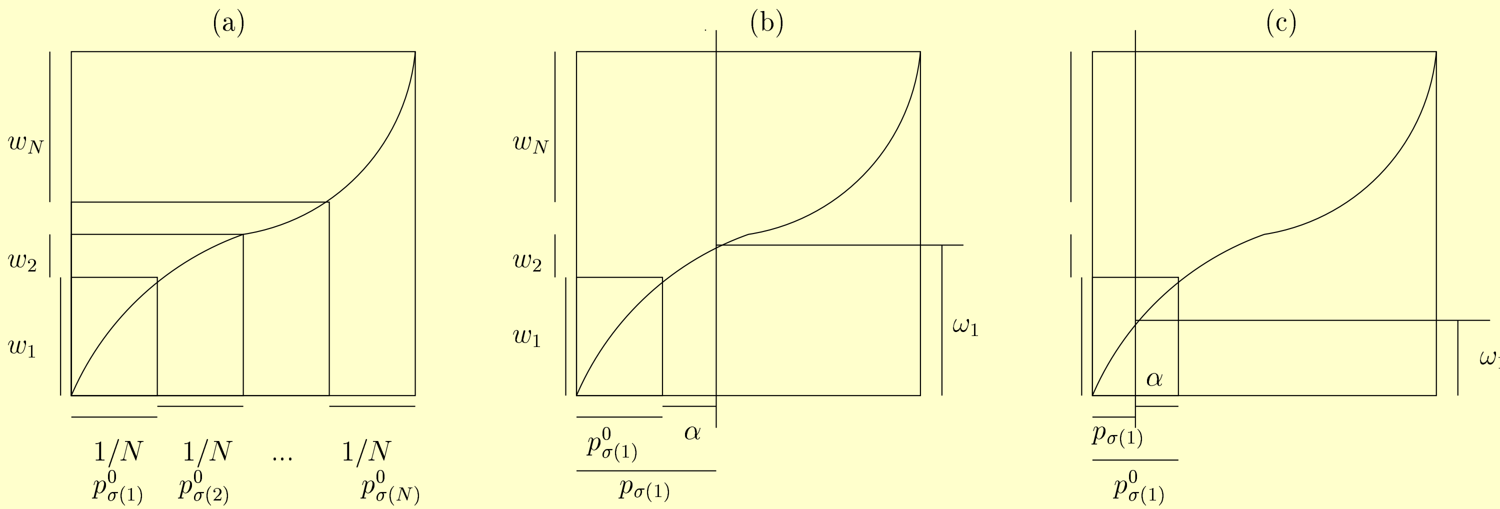
with σ a permutation of $\{1, ..., N\}$ s. t. $a_{\sigma(i-1)} \geq a_{\sigma(i)}$, and w^* a nondecreasing function that interpolates the points

$$\{(i/N, \sum_{j \leq i} w_j)\}_{i=1,...,N} \cup \{(0, 0)\}.$$

w^* is required to be a straight line when the points can be interpolated in this way.

Aggregation: WM, OWA, and WOWA operators

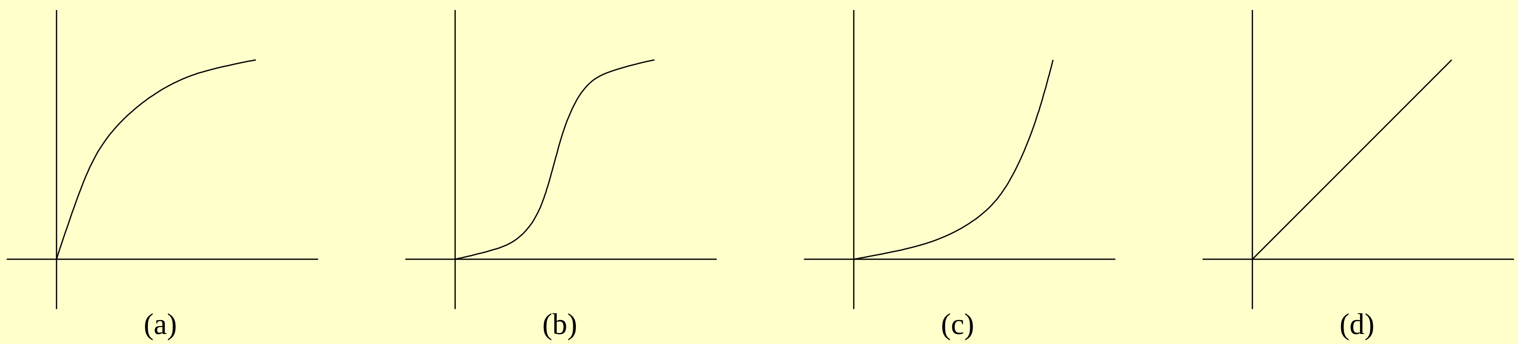
- Construction of the w^* quantifier



- Rationale for new weights (ω_i, for each value a_i) in terms of $\mathbf{p}$ and $\mathbf{w}$.
  - If a_i is small, and small values have more importance than larger ones, increase p_i for a_i (i.e., $\omega_i \geq p_{\sigma(i)}$).
    (the same holds if the value a_i is large and importance is given to large values)
  - If a_i is small, and importance is for large values, $\omega_i < p_{\sigma(i)}$
    (the same holds if a_i is large and importance is given to small values).

Aggregation: WM, OWA, and WOWA operators

- The shape of the function w^* gives importance
  - (a) to large values
  - (b) to medium values
  - (c) to small values
  - (d) equal importance to all values



Aggregation: WM, OWA, and WOWA operators

Example. Application

- Importance for constraints as given above: $\mathbf{p} = (0.5, 0.3, 0.15, 0.05)$
- Compensation as given above: $\mathbf{w} = (1/3, 1/3, 1/3, 0)$ (lowest value discarded)

  $\rightarrow$ WOWA with $\mathbf{p}$ and $\mathbf{w}$.

alternative	Aggregation of the Satisfaction degrees	WOWA
(x_A, x_B)	$WOWA_{\mathbf{p},\mathbf{w}}(C_1, C_2, C_3, C_4)$	
$(2, 2)$	$WOWA_{\mathbf{p},\mathbf{w}}(0, 0.5, 1, 1)$	0.4666
$(2, 3)$	$WOWA_{\mathbf{p},\mathbf{w}}(0.5, 0.5, 0.5, 0.5)$	0.5
$(2, 4)$	$WOWA_{\mathbf{p},\mathbf{w}}(1, 0.5, 0, 0.5)$	0.8333
$(3.5, 2.5)$	$WOWA_{\mathbf{p},\mathbf{w}}(1, 0.5, 0.5, 0.5)$	0.8333
$(3, 2)$	$WOWA_{\mathbf{p},\mathbf{w}}(0.5, 1, 1, 0.5)$	0.8
$(3, 3)$	$WOWA_{\mathbf{p},\mathbf{w}}(1, 1, 0.5, 1)$	1.0

Aggregation: WM, OWA, and WOWA operators

- Properties
  - The WOWA operator generalizes the WM and the OWA operator.
    - When $\mathbf{p} = (1/N \ \ldots \ 1/N)$, OWA

$$WOWA_{\mathbf{p},\mathbf{w}}(a_1, ..., a_N) = OWA_{\mathbf{w}}(a_1, ..., a_N) \text{ for all } \mathbf{w} \text{ and } a_i.$$

    - When $\mathbf{w} = (1/N \ ... \ 1/N)$, WM

$$WOWA_{\mathbf{p},\mathbf{w}}(a_1, ..., a_N) = WM_{\mathbf{p}}(a_1, ..., a_N) \text{ for all } \mathbf{p} \text{ and } a_i.$$

    - When $\mathbf{w} = \mathbf{p} = (1/N \ ... \ 1/N)$, AM

$$WOWA_{\mathbf{p},\mathbf{w}}(a_1, ..., a_N) = AM(a_1, ..., a_N)$$

Aggregation:

from the weighted mean to fuzzy integrals

Choquet integrals

Choquet integrals

- In WM, we combine a_i w.r.t. weights p_i.
  $\rightarrow$ a_i is the value supplied by information source x_i.
  Formally

Choquet integrals

- In WM, we combine a_i w.r.t. weights p_i.
  $\rightarrow$ a_i is the value supplied by information source x_i.
  Formally
  - $X = \{x_1, \ldots, x_N\}$ is the set of information sources
  - $f : X \rightarrow \mathbb{R}^+$ the values supplied by the sources
    $\rightarrow$ then $a_i = f(x_i)$

  Thus,

$$WM_{\mathbf{p}}(a_1, ..., a_N) = \sum_{i=1}^{N} p_i a_i = \sum_{i=1}^{N} p_i f(x_i) = WM_{\mathbf{p}}(f(x_1), ..., f(x_N))$$

Choquet integrals

- In the WM, a single weight is used for each element
  I.e., $p_i = p(x_i)$ (where, x_i is the information source that supplies a_i)
  $\rightarrow$ when we consider a set $A \subset X$, *weight* of$A$???

Choquet integrals

- In the WM, a single weight is used for each element
  I.e., $p_i = p(x_i)$ (where, x_i is the information source that supplies a_i)
  $\rightarrow$ when we consider a set $A \subset X$, *weight* of$A$???

... fuzzy measures $\mu(A)$

Formally,

– Fuzzy measure ($\mu : \wp(X) \rightarrow [0, 1]$), a set function satisfying
(i) $\mu(\emptyset) = 0$, $\mu(X) = 1$ (boundary conditions)
(ii) $A \subseteq B$ implies $\mu(A) \leq \mu(B)$ (monotonicity)

Choquet integrals

- Now, we have a fuzzy measure $\mu(A)$
  then, how aggregation proceeds?
  $\Rightarrow$ fuzzy integrals as the Choquet integral

Choquet integrals

- Choquet integral of f w.r.t. μ (alternative notation, $CI_\mu(a_1, \ldots, a_N)/CI_\mu(f)$)

$$(C)\int f d\mu = \sum_{i=1}^{N} [f(x_{s(i)}) - f(x_{s(i-1)})]\mu(A_{s(i)}),$$

where s in $f(x_{s(i)})$ is a permutation so that $f(x_{s(i-1)}) \leq f(x_{s(i)})$ for $i \geq 1$, $f(x_{s(0)}) = 0$, and $A_{s(k)} = \{x_{s(j)} | j \geq k\}$ and $A_{s(N+1)} = \emptyset$.

- Alternative expressions (Proposition 6.18):

$$(C)\int f d\mu = \sum_{i=1}^{N} f(x_{\sigma(i)})[\mu(A_{\sigma(i)}) - \mu(A_{\sigma(i-1)})],$$

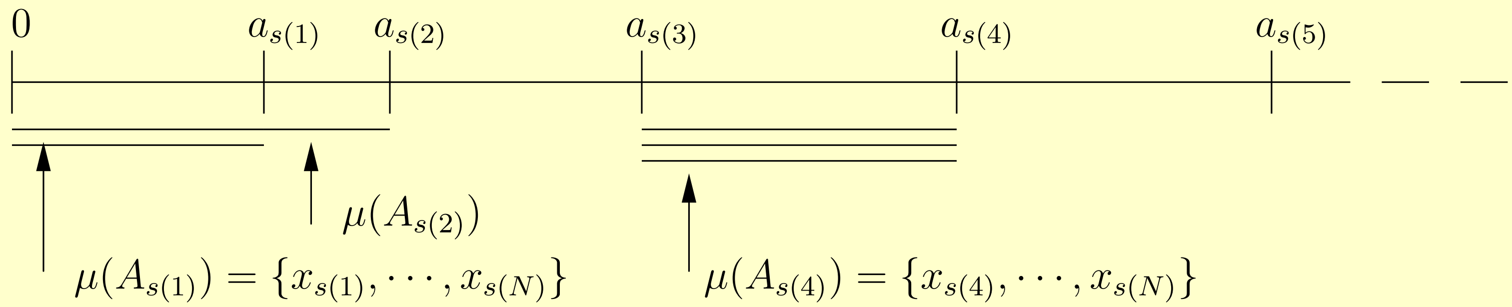
$$(C)\int f d\mu = \sum_{i=1}^{N} f(x_{s(i)})[\mu(A_{s(i)}) - \mu(A_{s(i+1)})],$$

where σ is a permutation of $\{1, \ldots, N\}$ s.t. $f(x_{\sigma(i-1)}) \geq f(x_{\sigma(i)})$, where $A_{\sigma(k)} = \{x_{\sigma(j)} | j \leq k\}$ for $k \geq 1$ and $A_{\sigma(0)} = \emptyset$

Choquet integrals

- Different equations point out different aspects of the CI

(6.1) $(C)\int f d\mu = \sum_{i=1}^{N}[f(x_{s(i)}) - f(x_{s(i-1)})]\mu(A_{s(i)})$,

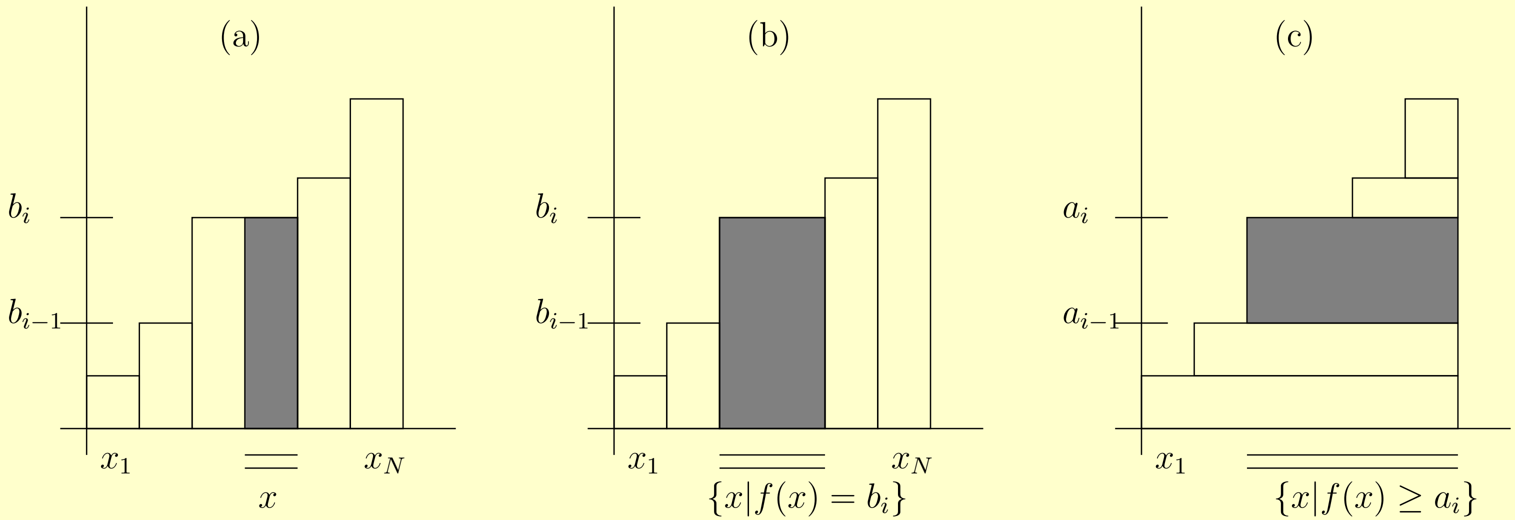


(6.2) $(C)\int f d\mu = \sum_{i=1}^{N} f(x_{\sigma(i)})[\mu(A_{\sigma(i)}) - \mu(A_{\sigma(i-1)})]$,

Choquet integrals

- $\int f d\mu =$ (for additive measures)

  (6.5) $\sum_{x \in X} f(x)\mu(\{x\})$
  (6.6) $\sum_{i=1}^{R} b_i \mu(\{x | f(x) = b_i\})$
  (6.7) $\sum_{i=1}^{N} (a_i - a_{i-1}) \mu(\{x | f(x) \geq a_i\})$
  (6.8) $\sum_{i=1}^{N} (a_i - a_{i-1}) \big(1 - \mu(\{x | f(x) \leq a_{i-1}\})\big)$

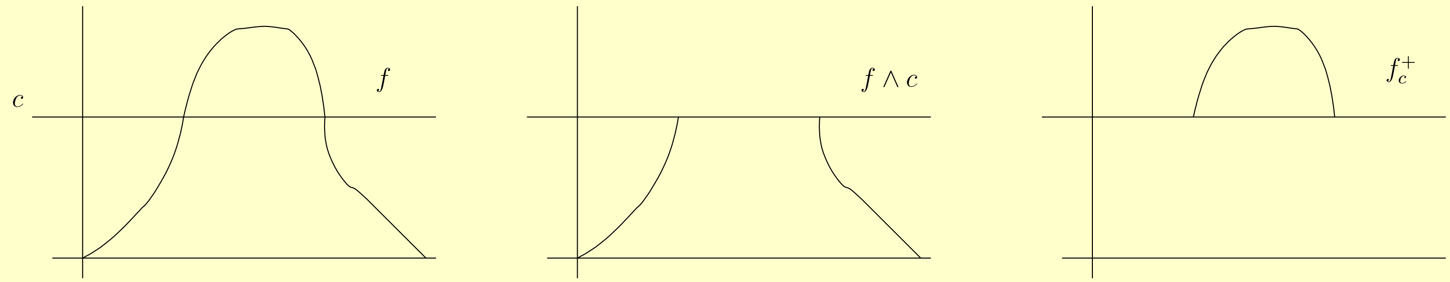


- Among (6.5), (6.6) and (6.7), only (6.7) satisfies internality.

Choquet integrals

- Properties of CI
  - Horizontal additive because $CI_\mu(f) = CI_\mu(f \wedge c) + CI_\mu(f_c^+)$
    ($f = (f \wedge c) + f_c^+$ is a horizontal additive decomposition of f)
    where, f_c^+ is defined by (for $c \in [0,1]$)

$$f_c^+ = \begin{cases} 0 & \text{if } f(x) \leq c \\ f(x) - c & \text{if } f(x) > c. \end{cases}$$



Choquet integrals

- Definitions (X a reference set, f, g functions $f, g : X \to [0, 1]$)
  - $f < g$ when, for all x_i,
  $$f(x_i) < g(x_i)$$
  - f and g are comonotonic if, for all $x_i, x_j \in X$,
  $$f(x_i) < f(x_j) \text{ imply that } g(x_i) \leq g(x_j)$$
  - $\mathbb{C}$ is comonotonic monotone if and only if, for comonotonic f and g,
  $$f \leq g \text{ imply that } \mathbb{C}(f) \leq \mathbb{C}(g)$$
  - $\mathbb{C}$ is comonotonic additive if and only if, for comonotonic f and g,
  $$\mathbb{C}(f + g) = \mathbb{C}(f) + \mathbb{C}(g)$$

- Characterization. Let $\mathbb{C}$ satisfy the following properties
  - $\mathbb{C}$ is comonotonic monotone
  - $\mathbb{C}$ is comonotonic additive
  - $\mathbb{C}(1, \ldots, 1) = 1$

  Then, there exists μ s.t. $\mathbb{C}(f)$ is the CI of f w.r.t. μ.

Choquet integrals

- Properties
  - WM, OWA and WOWA are particular cases of CI.
    - WM with weighting vector $\mathbf{p}$ is a CI w.r.t. $\mu_{\mathbf{p}}(B) = \sum_{x_i \in B} p_i$
    - OWA with weighting vector $\mathbf{w}$ is a CI w.r.t. $\mu_{\mathbf{w}}(B) = \sum_{i=1}^{|B|} w_i$
    - WOWA with w.v. $\mathbf{p}$ and $\mathbf{w}$ is a CI w.r.t. $\mu_{\mathbf{p},\mathbf{w}}(B) = w^*(\sum_{x_i \in B} p_i)$
  - Any symmetric CI is an OWA operator.
  - Any CI with a distorted probability is a WOWA operator.
  - Let A be a crisp subset of X; then, the Choquet integral of A with respect to μ is $\mu(A)$.

    Here, the integral of A corresponds to the integral of its characteristic function, or, in other words, to the integral of the function f_A defined as $f_A(x) = 1$ if and only if $x \in A$.

Aggregation:

from the weighted mean to fuzzy integrals

Weighted minimum and maximum

Weighted Minimum and Weighted Maximum

- Possibilistic weighting vector (dimension N): $\mathbf{v} = (v_1 ... v_N)$ iff $v_i \in [0,1]$ and $\max_i v_i = 1$.

- Weighted minimum (WMin: $[0,1]^N \to [0,1]$):
$WMin_{\mathbf{u}}(a_1, ..., a_N) = \min_i \max(neg(u_i), a_i)$
(alternative definition can be given with $\mathbf{v} = (v_1, \ldots, v_N)$ where $v_i = neg(u_i)$)

- Weighted maximum (WMax: $[0,1]^N \to [0,1]$):
$WMax_{\mathbf{u}}(a_1, ..., a_N) = \max_i \min(u_i, a_i)$

Weighted Minimum and Weighted Maximum

- **Exemple 6.34.** Evaluation of the alternatives related to the course
  - Weighting vector (possibilistic vector): $\mathbf{u} = (1,\ 0.5,\ 0.3,\ 0.1)$.
  - WMin:
    * $sat(2,2) = WMin_{\mathbf{u}}(0,\ 0.5,\ 1,\ 1) = 0$
    * $sat(2,3) = WMin_{\mathbf{u}}(0.5,\ 0.5,\ 0.5,\ 0.5) = 0.5$
    * $sat(2,4) = WMin_{\mathbf{u}}(1,\ 0.5,\ 0,\ 0.5) = 0.5$
    * $sat(3.5,\ 2.5) = WMin_{\mathbf{u}}(1,\ 0.5,\ 0.5,\ 0.5) = 0.5$
    * $sat(3,2) = WMin_{\mathbf{u}}(0.5,\ 1,\ 1,\ 0.5) = 0.5$
    * $sat(3,3) = WMin_{\mathbf{u}}(1,\ 1,\ 0.5,\ 1) = 0.7.$
  - WMax: (with $neg(\mathbf{u}) = (0,\ 0.5,\ 0.7,\ 0.9)$, using $neg(x) = 1 - x$)
    * $sat(2,2) = WMax_{\mathbf{u}}(0,\ 0.5,\ 1,\ 1) = 0.5$
    * $sat(2,3) = WMax_{\mathbf{u}}(0.5,\ 0.5,\ 0.5,\ 0.5) = 0.5$
    * $sat(2,4) = WMax_{\mathbf{u}}(1,\ 0.5,\ 0,\ 0.5) = 1$
    * $sat(3.5,\ 2.5) = WMax_{\mathbf{u}}(1,\ 0.5,\ 0.5,\ 0.5) = 1$
    * $sat(3,2) = WMax_{\mathbf{u}}(0.5,\ 1,\ 1,\ 0.5) = 0.5$
    * $sat(3,3) = WMax_{\mathbf{u}}(1,\ 1,\ 0.5,\ 1) = 1.$
  - weighted minimum, the best pair is $(3,3)$; with weighted maximum $(3,3)$, $(2,4)$ and $(3,5,\ 2,5)$ indistinguishable

Weighted Minimum and Weighted Maximum

- **Exemple 6.35.** Fuzzy inference system
  R_i: **IF** x is A_i **THEN** y is B_i.
  - with disjunctive rules, the (fuzzy) output for a particular y_0 is a WMax

$$\tilde{B}(y_0) = \vee_{i=1}^{N}\big(B_i(y_0) \wedge A_i(x_0)\big).$$

  - with conjunctive rules, and Kleene-Dienes implication $(\mathcal{I}(x,y) = \max(1-x,y))$ the (fuzzy) output of the system for a particular y_0 is a WMin

$$\tilde{B}(y_0) = \wedge_{i=1}^{N}\big(\mathcal{I}(A_i(x_0), B_i(y_0))\big) = \wedge_{i=1}^{N} \max(1 - A_i(x_0), B_i(y_0)).$$

  that with $\mathbf{u} = (A_1(x_0), \ldots, A_N(x_0))$

$$\tilde{B}(y_0) = WMin_{\mathbf{u}}(B_1(y_0), \ldots, B_N(y_0)).$$

Weighted Minimum and Weighted Maximum

- Only operators in ordinal scales (max, min, neg) are used in $WMax$ and $WMin$.
- neg is completely determined in an ordinal scale

Proposition 6.36. Let $L = \{l_0, \ldots, l_r\}$ with $l_0 <_L l_1 <_L \cdots <_L l_r$; then, there exists only one function, $neg : L \to L$, satisfying

(N1) if $x <_L x'$ then $neg(x) >_L neg(x')$ for all x, x' in L.
(N2) $neg(neg(x)) = x$ for all x in L.

This function is defined by $neg(x_i) = x_{r-i}$ for all x_i in L

- Properties. For $\mathbf{u} = (1, \ldots, 1)$
  - $WMIN_{\mathbf{u}} = \min$
  - $WMAX_{\mathbf{u}} = \max$

Aggregation:

from the weighted mean to fuzzy integrals

Sugeno integral

Sugeno integral

- Sugeno integral of f w.r.t. μ (alternative notation, $SI_\mu(a_1, \ldots, a_N)/SI_\mu(f)$)

$$(S)\int f d\mu = \max_{i=1,N} \min(f(x_{s(i)}), \mu(A_{s(i)})),$$

where s in $f(x_{s(i)})$ is a permutation so that $f(x_{s(i-1)}) \leq f(x_{s(i)})$ for $i \geq 2$, and $A_{s(k)} = \{x_{s(j)} | j \geq k\}$.

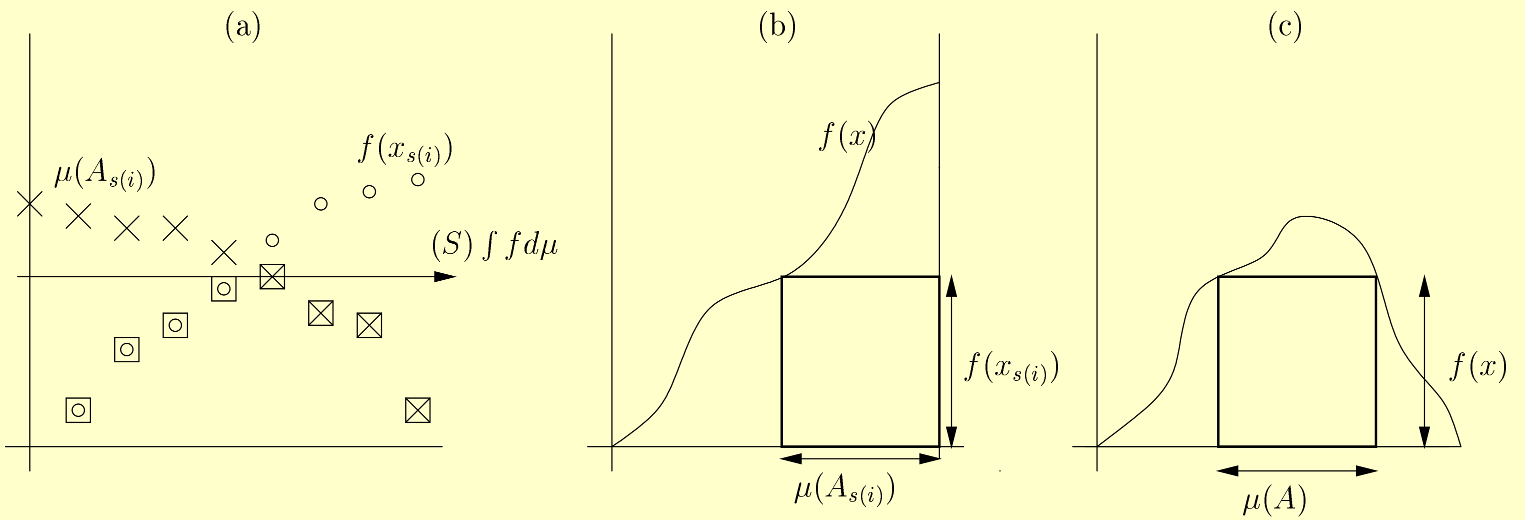
- Alternative expression (Proposition 6.38):

$$\max_i \min(f(x_{\sigma(i)}), \mu(A_{\sigma(i)})),$$

where σ is a permutation of $\{1, \ldots, N\}$ s.t. $f(x_{\sigma(i-1)}) \geq f(x_{\sigma(i)})$, where $A_{\sigma(k)} = \{x_{\sigma(j)} | j \leq k\}$ for $k \geq 1$

Sugeno integral

- Graphical interpretation of Sugeno integrals



Sugeno integral

- Properties
  - WMin and WMax are particular cases of SI
    - WMax with weighting vector $\mathbf{u}$ is a SI w.r.t. $\mu^{wmax}_{\mathbf{u}}(A) = \max_{a_i \in A} u_i$.
    - WMin with weighting vector $\mathbf{u}$ is a SI w.r.t. $\mu^{wmin}_{\mathbf{u}}(A) = 1 - \max_{a_i \notin A} u_i$.

Sugeno integral

Example. Citation indices

- Number of citations: CI
- h-index: SI

In both cases,

- X the set of papers
- $f(x)$ the number of citations of paper x
- $\mu(A) \subseteq X$ the cardinality of the set

Fuzzy integrals

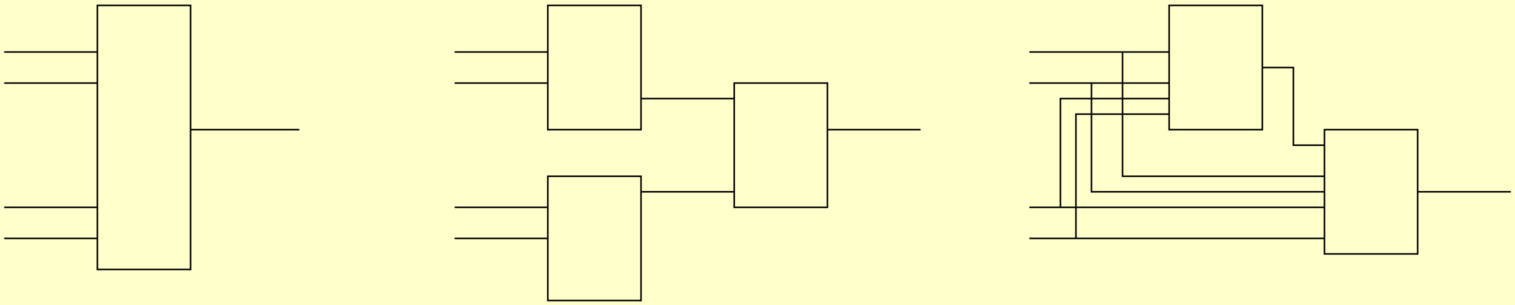
- Fuzzy integrals that generalize Choquet and Sugeno integrals
  - The fuzzy t-conorm integral
  - The twofold integral

Aggregation:

Hierarchical models

Hierarchical Models for Aggregation

- Hierarchical model



- Properties. The following conditions hold

  (i) Every multistep Choquet integral is a monotone increasing, positively homogeneous, piecewise linear function.

  (ii) Every monotone increasing, positively homogeneous, piecewise linear function on a full-dimensional convex set in $\mathbb{R}^N$ is representable as a two-step Choquet integral such that the fuzzy measures of the first step are additive and the fuzzy measure of the second step is a 0-1 fuzzy measure.

Aggregation for preference relations

(MCDM: social choice)

Aggregation for preference relations

- MCDM (decision) and social choice
  $\Rightarrow$ are two related areas

Aggregation for preference relations

- Social choice
  - studies voting rules, and how the preferences of a set of people can be aggregated to obtain the preference of the set.
- There is no formal difference between aggregation of opinions from people and aggregation of criteria

Aggregation for preference relations

- Given preference relations, how aggregation is built?
  - Formalization of preferences with $>$ an $=$ (preference, indiference)
  - $F(R_1, R_2, \ldots, R_N)$ to denote aggregated preference

Aggregation for preference relations

- Given preference relations, how aggregation is built?
  - Formalization of preferences with $>$ an $=$ (preference, indiference)
  - $F(R_1, R_2, \ldots, R_N)$ to denote aggregated preference
    - Problems (I): consider
      * $R^1 : x > y > z$
      * $R^4 : y > z > x$
      * $R^5 : z > x > y$

      $\rightarrow$ simple majority rule: $u > v$ if most prefer u to v
      * $x > y, y > z, z > x$ (intransitive!!: $x > y, y > z$ but not $x > z$)
    - Problems (II):

      $\rightarrow$ Arrow impossibility theorem

Aggregation for preference relations

- Given preference relations, how aggregation is built?
- Axioms of Arrow impossibility theorem

  C0 Finite number of voters and more than one
  Number of alternatives more or equal to three

  C1 Universality: Voters can select any total preorder

  C2 Transitivity: The result is a total preorder

  C3 Unanimity: If all agree on x better than y, then x better than y in the social preference

  C4 Independence of irrellevant alternatives: the social preference of x and y only depends on the preferences on x and y

  C5 No-dictatorship: No voter can be a dictatorship

- There is no function F that satisfies all C0-C5 axioms

Aggregation for preference relations

- Given preference relations, how aggregation is built?
- Circumventing Arrow's theorem
  - Ignore the condition of universality
  - Ignore the condition of independence of irrelevant alternatives

Aggregation for preference relations

- Given preference relations, how aggregation is built?
  - Solutions failing the universality condition
    - Simple peak, odd number of voters,
      Condorcet rule satisfies all other conditions

Aggregation for preference relations

- Given preference relations, how aggregation is built?
  - Solutions failing the condition of independence of irrelevant alternatives
    * Condorcet tule with Copeland[1]:
    * Borda count[2]

[1] Defined by Ramon Llull s. XIII

[2] Defined by Nicolas de Cusa s. XV.

Related topics

Related topics

- Aggregation functions
  - Functional equations (synthesis of judgements)
  - Fuzzy measures
  - Indices and evaluation methods
  - Model selection
- Decision making
  - Game theory (for decision making with adversary)
  - Decision under risk and uncertainty
  - Voting systems (social choice, aggregation of preferences)

Thank you