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On Distorted Probabilities and Other Fuzzy Measures

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Introduction

- Fuzzy measures.
  - Application in information fusion / aggregation
    $\longrightarrow$ fuzzy integrals
  - Used to represent *background knowledge* in fuzzy integrals

Outline

- Introduction
- Distorted Probabilities
- Extension of Distorted Probabilities
- m-dimensional Distorted Probabilities
- Distorted Probabilities and OWA
- Conclusions

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Introduction

- Running example
  - A toy example that provides a framework for application of fuzzy measures and aggregation
- A few definitions
  - Aggregation operators
  - Fuzzy measures

Introduction

- Running example (originally, from Grabisch):
  - From marks on mathematics (M), physics (P) and literature (L) calculate an *overall mark* (all marks on $[0, 1]$) for a set of students.

Student	M	P	L	Overall Mark
Nobita	0	0	1	?
Alfred	0	1	0	?
Ezequiel	1	0	0	?
Arare	1	1	0	?
Joan	1	0	1	?
Berta	1	1	1	?

- Then, we can rank the students or select *the best*.

Example

- Running example (originally, from Grabisch):
  - From marks on mathematics (M), physics (P) and literature (L) calculate an *overall mark* (all marks on $[0, 1]$) for a set of students.
  - Standard approach:
    - Compute an overall mark
      $\longrightarrow$ using an aggregation operator
      (often, a parametric aggregation operator)

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  - From marks on mathematics (M), physics (P) and literature (L) calculate an *overall mark* (all marks on $[0, 1]$) for a set of students.
  - Computation of the overall mark: aggregation
    - ∗ Through the weighted mean
      - · We need to define the weights of the subjects

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      - We need to define the weights of the subjects
    - Through a fuzzy integral
      - We need to define the measure on the set of subjects

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  - From marks on mathematics (M), physics (P) and literature (L) calculate an *overall mark* (all marks on $[0,1]$) for a set of students.
  - Computation of the overall mark: aggregation
    - Through the weighted mean
      - We need to define the weights of the subjects $X = \{P, M, L\}$, $p : X \to [0,1]$ with some constraints on p
    - Through a fuzzy integral
      - We need to define the measure on the set of subjects $X = \{P, M, L\}$, $\mu : 2^X \to [0,1]$ with some constraints on μ

Example

- Running example: defining a fuzzy measure

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1. Boundary conditions:

   If all marks zero, final mark equals zero;
   If all maximum marks, final mark equals maximum mark.
2. Relative importance of scientific versus literary subjects:

   The importance of mathematics and physics is greater than the importance of literature.
3. *Redudancy* between mathematics and physics:

   Mathematics and physics are similar subjects. The importance of the set containing both should not be larger than their addition.
4. Support between literature and scientific subjects:

   Mathematics and literature are complementary subjects.

Example

- Running example: defining a fuzzy measure

1. Boundary conditions:
   $\mu(\emptyset) = 0$, $\mu(\{M, P, L\}) = 1$
   The importance of the empty set is 0.
   The set consisting of all objects has maximum importance.
2. Relative importance of scientific versus literary subjects:
   $\mu(\{M\}) = \mu(\{P\}) = 0.45$, $\mu(\{L\}) = 0.3$
   The importance of mathematics and physics is greater than the importance of literature.
3. *Redudancy* between mathematics and physics:
   $\mu(\{M, P\}) = 0.5 < \mu(\{M\}) + \mu(\{P\})$
   Mathematics and physics are similar subjects. The importance of the set containing both should not be larger than their addition.
4. Support between literature and scientific subjects:
   $\mu(\{M, L\}) = \mu(\{P, L\}) = 0.9 > \mu(\{P\}) + \mu(\{L\}) = 0.45 + 0.3 = 0.75$
   $\mu(\{M, L\}) = \mu(\{P, L\}) = 0.9 > \mu(\{M\}) + \mu(\{L\}) = 0.45 + 0.3 = 0.75$
   Mathematics and literature are complementary subjects.

Example

- Running example: defining weights

1. Boundary conditions:
   (always true for the weighted mean)
2. Relative importance of scientific versus literary subjects:
   $\mu(\{M\}) = \mu(\{P\}) = 0.45$, $\mu(\{L\}) = 0.3$
   $\rightarrow p(M) = p(P) = 0.45/1.2 = 0.375$, $p(L) = 0.3/1.2 = 0.25$
   The importance of mathematics and physics is greater than the importance of literature.

Definitions (I)

- Formalization of the aggregation process: aggregation operators

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  - Choquet integral of f w.r.t. μ is

$$(C)\int f d\mu = \sum_{i=1}^{N} f(x_{\sigma(i)})[\mu(A_{\sigma(i)}) - \mu(A_{\sigma(i-1)})]$$

  - Sugeno integral of f w.r.t. μ is

$$(S)\int f d\mu = \max_{i=1,N} \min(f(x_{s(i)}), \mu(A_{s(i)}))$$

σ a permutation of $\{1, ..., N\}$ s.t. $a_{\sigma(i-1)} \geq a_{\sigma(i)}$ for all $i \geq 2$, s a permutation s. t. $a_{s(i-1)} \leq a_{s(i)}$, $A_{\sigma(k)} = \{x_{\sigma(j)} | j \leq k\}$, $A_{s(i)} = \{x_{s(j)} | i \geq i\}$.

Example

- Running example:
  - Weighted mean:

    $p(L) = 0.3/1.2 = 0.25$, $p(P) = 0.45/1.2 = 0.375$, $p(M) = 0.45/1.2 = 0.375$
  - Choquet integral:

    $\mu(\emptyset) = 0$ $\quad \mu(\{M, L\}) = 0.9$

    $\mu(\{M\}) = 0.45$ $\quad \mu(\{P, L\}) = 0.9$

    $\mu(\{P\}) = 0.45$ $\quad \mu(\{M, P\}) = 0.5$

    $\mu(\{L\}) = 0.3$ $\quad \mu(\{M, P, L\}) = 1$
  - Overall mark:

Student	M	P	L	WM	CI
Nobita	0	0	1	0.25	0.3
Alfred	0	1	0	0.375	0.45
Ezequiel	1	0	0	0.375	0.45
Arare	1	1	0	0.75	0.45
Joan	1	0	1	0.625	0.9
Berta	1	1	1	1.0	1.0

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  - $\mu : 2^X \rightarrow [0, 1]$ is a fuzzy measure:

    (i) $\mu(\emptyset) = 0$, $\mu(X) = 1$ (boundary conditions)

    (ii) $A \subseteq B$ implies $\mu(A) \leq \mu(B)$ (monotonicity)

Definitions (II)

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    (i) $\mu(\emptyset) = 0$, $\mu(X) = 1$ (boundary conditions)
    (ii) $A \subseteq B$ implies $\mu(A) \leq \mu(B)$ (monotonicity)
  - m is the Möbius transform of μ ($A \subseteq X$):

$$m_\mu(A) := \sum_{B \subseteq A} (-1)^{|A|-|B|} \mu(B)$$

Definitions (III)

- Formalization of the aggregation process: fuzzy measures
  - Unconstrained fuzzy measures require $2^{|X|} - 2$ parameters
- Families of fuzzy measures (with reduced complexity):
  - Decomposable fuzzy measures:
    $\mu(A \cup B) = \mu(A) \perp \mu(B)$ for $A \cap B = \emptyset$
  - k-additive fuzzy measures:
    $m(A) = 0$ for all $|A| > k$
  - m-symmetric fuzzy measures

Definitions (IV)

- Formalization of the aggregation process: fuzzy measures
- A key property of k-additive fuzzy measures.
  - $KAFM_{k_0,X}$, the set of all k_0-additive fuzzy measures.
  - Then,

$$KAFM_{1,X} \subset KAFM_{2,X} \subset KAFM_{3,X} \cdots \subset KAFM_{|X|,X}$$

  - It follows that:
    1. $KAFM_{1,X}$ equals the set of additive measures
    2. $KAFM_{|X|,X}$ equals the set of all fuzzy measures over X
       ($KAFM_{|X|,X}$ is the set of unconstrained fuzzy measures)

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- This family covers the whole set of fuzzy measures

- The smaller k_0, the less parameters are required

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Distorted Probabilities

- Representation of a fuzzy measure:
  - f and P represent a fuzzy measure μ, iff

$$\mu(A) = f(P(A)) \text{ for all } A \in 2^X$$

    f a real-valued function, P a probability measure on $(X, 2^X)$
  - f is strictly increasing w.r.t. a probability measure P iff $P(A) < P(B)$ implies $f(P(A)) < f(P(B))$
  - f is nondecreasing w.r.t. a probability measure P iff $P(A) < P(B)$ implies $f(P(A)) \leq f(P(B))$

Distorted Probabilities

- Representation of a fuzzy measure: distorted probability
  - f and P represent a fuzzy measure μ, iff

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  - μ is a distorted probability if μ is represented by a probability distribution P and a function f nondecreasing w.r.t. a probability P.

Distorted Probabilities

- Representation of a fuzzy measure: distorted probability[1]
  - μ is a distorted probability if μ is represented by a probability distribution P and a function f nondecreasing w.r.t. a probability P
- So, for a given reference set X we need:
  - Probability distribution on X: $p(x)$ for all $x \in X$
  - Distortion function f on the probability measure: $f(P(A))$

    $\rightarrow$ as μ is a measure: $f(0) = 0$, $f(1) = 1$

    $\rightarrow$ f a nondecreasing fuzzy quantifier

[1] Suggested by Edwards (1953) in experimental psycology

Application

- Given a distorted probability ...
  ... we can apply any fuzzy integral

- E.g.
  - the Choquet integral
  - the Sugeno integral

Properties

- Distorted probability and Choquet integral:
  - The WOWA operator can be represented by a Choquet integral with a distorted probability.
    - WOWA generalizes both the WM and the OWA, using both WM weights and OWA weights.
  - From the distorted probability perspective, in WOWA:
    - the WM weights correspond to the probability distribution
    - the OWA weights are used to build the distortion function (the nondecreasing fuzzy quantifier)

Properties

- Trying to find a characterization ...
  - pre-distorted probability: If $\mu(A) < \mu(B) \Leftrightarrow \mu(A \cup C) < \mu(B \cup C)$ for every $A \cap C = \emptyset$, $B \cap C = \emptyset$ $A, B, C \in 2^X$,
- Distorted probs. (with strictly increasing f) are pre-distorted probabilities (similar result with nondecreasing f: $\mu(A) < \mu(B)$ implies $\mu(A \cup C) \leq \mu(B \cup C)$)
- However, the reversal is not true in general. (it is true for $|X| = 3$ and $|X| = 4$ when $\mu(A) \neq \mu(B)$ for all $A \neq B$)

Properties

- Not all fuzzy measures are distorted probabilities
- Nevertheless,
  - all fuzzy measures are representable by a polynomial f and a probability P as $\mu = f \circ P$.

    (nondecreasingness of f can not be guaranteed)
  - any fuzzy measure can be represented as the difference of two distorted probabilities:

$$\mu = f^{+} \circ P - f^{-} \circ P$$

    (f^{+} and f^{-} are strictly increasing polynomials)

Properties

- Some relationships with other measures.
- Distorted probabilities generalize
  - $\perp$-decomposable fuzzy measures
  - Sugeno λ-measures
    - if for some fixed $\lambda > -1$, it holds for all $A \cap B = \emptyset$

$$\mu(A \cup B) = \mu(A) + \mu(B) + \lambda \mu(A) \mu(B)$$

Properties

- Some distorted probabilities are not decomposable fuzzy measures.
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Properties

- Some distorted probabilities are not decomposable fuzzy measures.
- Some distorted probabilities cannot be represented easily with other families of fuzzy measures.
- 1st. example (I):
  - μ on $X = \{a, b, c\}$ with $p(a) = 0.2$, $p(b) = 0.35$, $p(c) = 0.45$, and

$$f(x) = \begin{cases} 0 & \text{if } x < 0.5 \\ 0.2 & \text{if } 0.5 \leq x < 0.6 \\ 0.4 & \text{if } 0.6 \leq x < 0.85 \\ 1.0 & \text{if } 0.85 \leq x \leq 1.0 \end{cases}$$

Properties

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- 1st. example (II):
  - $\mu(\emptyset) = 0$, $\mu(\{a\}) = 0$, $\mu(\{b\}) = 0$, $\mu(\{c\}) = 0$,
    $\mu(\{a,b\}) = 0.2$, $\mu(\{a,c\}) = 0.4, \mu(\{b,c\}) = 0.4, \mu(\{a,b,c\}) = 1$
  - μ is a DP but not a $\perp$-decomposable fuzzy measure
    because, there is no t-conorm s.t. $\perp(0,0) \neq 0$
    $\rightarrow$ as $\mu(\{a,b\}) = 0.2$ when $\mu(\{a\}) = 0$ and $\mu(\{b\}) = 0$,
    we would require $0.2 = \mu(\{a,b\}) = \perp(\mu(\{a\}), \mu(\{b\})) = \perp(0,0)$.

Properties

- Some distorted probabilities are not decomposable fuzzy measures.
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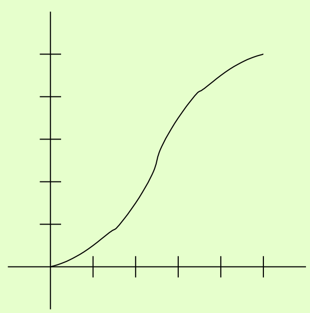
- 2nd. example (I):
  - $\mu_{\mathbf{p},\mathbf{w}}$ over $X = \{x_1, x_2, x_3, x_4, x_5\}$ from (probability distribution) $\mathbf{p} = (0.2, 0.3, 0.1, 0.2, 0.1)$, and function (from $\mathbf{w} = (0.1, 0.2, 0.4, 0.2, 0.1)$):



Properties

- Some distorted probabilities are not decomposable fuzzy measures.
- Some distorted probabilities cannot be represented easily with other families of fuzzy measures.
- 2nd. example (II):
  - $\mu_{\mathbf{p},\mathbf{w}}$ is a 5-additive fuzzy measure because $m(A) \neq 0$ for all A.
  - E.g.,
    $m(\{x_1, x_2, x_3, x_4, x_5\}) = 0.50746528$,
    $m(\{x_1, x_2, x_3, x_4\}) = -0.2537326$.
    There is no k-additive fuzzy measure equivalent to $\mu_{\mathbf{p},\mathbf{w}}$ for $k < 5$.

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Extension of Distorted Probabilities: Why?

- Question:
  How many fuzzy measures can be represented by a distorted probability?

Motivation

- Counting the number of distorted probabilities.
  - Assume $X = \{1, 2, 3\}$.
  - Assume μ with $\mu(\{1\}) < \mu(\{2\}) < \mu(\{3\})$.
  - Possible distorted probabilities:

$$\mu(\emptyset) < \mu(\{1\}) < \mu(\{2\}) < \mu(\{3\}) < \mu(\{1,2\}) < \mu(\{1,3\}) < \mu(\{2,3\}) < \mu(X)$$
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- Constraints for measures $\mu(A) \leq \mu(B)$ if $A \subseteq B$

- Constraints due to probabilities:
  $p(1) < p(2) < p(3)$ implies $p(1) + p(3) < p(1) + p(2)$
  therefore, $\mu(\{1,3\}) < \mu(\{1,2\})$

Motivation

- Counting the number of distorted probabilities.
  - Assume $X = \{1, 2, 3\}$.
  - Assume μ with $\mu(\{1\}) < \mu(\{2\}) < \mu(\{3\})$.
  - Possible unconstrained fuzzy measures:

$$\emptyset < \mu(\{1\}) < \mu(\{2\}) < \mu(\{3\}) < \mu(\{1,2\}) < \mu(\{1,3\}) < \mu(\{2,3\}) < \mu(X)$$
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Motivation

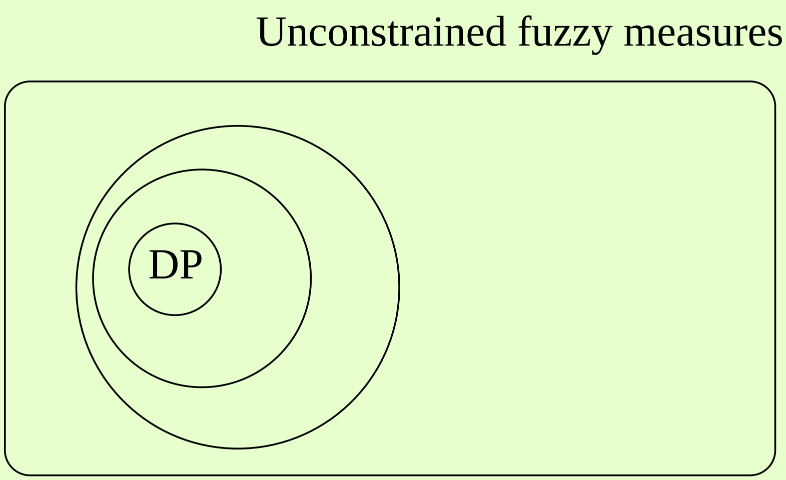
- Counting the number of distorted probabilities.
  - For $X = \{1, 2, 3\}$, 2/8 of distorted probabilities.
  - For larger sets X ...

    ... the proportion of distorted probabilities decreases rapidly
  - For $\mu(\{1\}) \leq \mu(\{2\}) \leq \ldots$

$\|X\|$	Number of possible orderings for Distorted Probabilities	Number of possible orderings for Fuzzy Measures
1	1	1
2	1	1
3	2	8
4	14	70016
5	546	$\mathcal{O}(10^{12})$
6	215470	–

Motivation

- Goal:
  - Cover a larger region of the space of fuzzy measures



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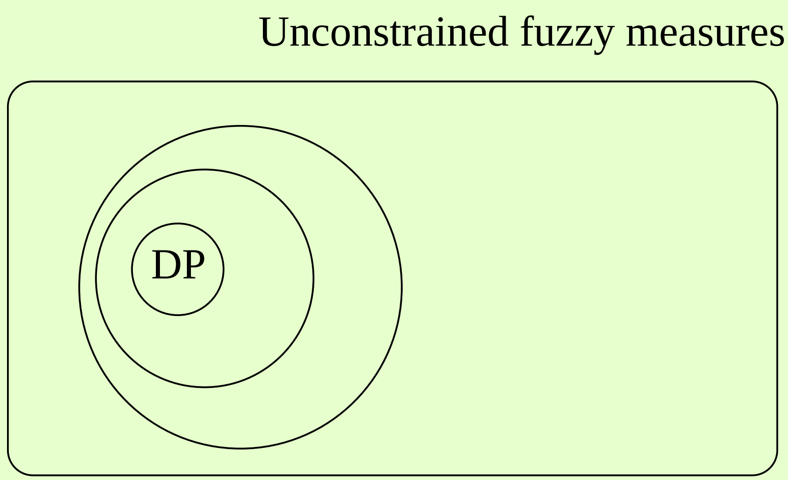

$\rightarrow$ Similar to the property of k-additive fuzzy measures:

$$KAFM_{1,X} \subset KAFM_{2,X} \subset KAFM_{3,X} \cdots \subset KAFM_{|X|,X}$$

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m-dimensional Distorted Probabilities

- In distorted probabilities:
  - One probability distribution
  - One function f to distort the probabilities
- Extension to:
  - m probability distributions
  - One function f to distort/combine the probabilities

m-dimensional Distorted Probabilities

- In distorted probabilities:
  - One probability distribution
  - One function f to distort the probabilities

- Extension to:
  - m probability distributions P_i
    - Each P_i defined on X_i
    - Each X_i is a partition element of X (a dimension)
  - One function f to distort/combine the probabilities

Example

- Running example:
  - Reconsidering the fuzzy measure:

$\mu(\emptyset) = 0$	$\mu(\{M, L\}) = 0.9$
$\mu(\{M\}) = 0.45$	$\mu(\{P, L\}) = 0.9$
$\mu(\{P\}) = 0.45$	$\mu(\{M, P\}) = 0.5$
$\mu(\{L\}) = 0.3$	$\mu(\{M, P, L\}) = 1$

  - Partition on X:
    * $X_1 = \{L\}$ (Literary subjects)
    * $X_2 = \{M, P\}$ (Scientific Subjects)

Definition

- m-dimensional distorted probabilities.
  - μ is an at most m dimensional distorted probability if

$$\mu(A) = f(P_1(A \cap X_1), P_2(A \cap X_2), \cdots, P_m(A \cap X_m))$$

where,
$\{X_1, X_2, \cdots, X_m\}$ is a partition of X,
P_i are probabilities on $(X_i, 2^{X_i})$,
f is a function on $\mathbb{R}^m$ strictly increasing with respect to the i-th axis for all $i = 1, 2, \ldots, m$.

- μ is an m-dimensional distorted probability if it is an at most m dimensional distorted probability but it is not an at most $m-1$ dimensional.

Example

- Running example: a two dimensional distorted probability

$$\mu(A) = f(P_1(A \cap \{L\}), P_2(A \cap \{M, P\}))$$

  - with partition on $X = \{M, L, P\}$
    1. Literary subject $\{L\}$
    2. Science subjects $\{M, P\}$,
  - probabilities
    1. $P_1(\{L\}) = 1$
    2. $P_2(\{M\}) = P_2(\{P\}) = 0.5$,
  - and distortion function f defined by

1	{L}	0.3	0.9	1.0
0	∅	0	0.45	0.5
	sets	∅	{M}, {P}	{M,P}
f		∅	0.5	1

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Distorted Probabilities and OWA

- OWA operators and Choquet integrals:
  - OWA equivalent to a Choquet integral w.r.t. a symmetric μ
- μ is symmetric when (informally, only cardinality is important)

$$\mu(A) = \mu(B), \text{ if } |A| = |B|$$

- Miranda et al. introduced m-symmetric fuzzy measures.

m-symmetric fuzzy measures

- m-symmetric fuzzy measures (Miranda et al., 2002):
  - $A \subseteq X$ is a set of indifference iff (informally, only cardinality is important)

$$\forall B_1, B_2 \subseteq A, |B_1| = |B_2|,$$

$$\forall C \subseteq X \setminus A \quad \mu(B_1 \cup C) = \mu(B_2 \cup C)$$

  - 2-symmetric fuzzy measure μ iff:
    * there exists a partition of $\{X_1, X_2\}$ $(X_1, X_2 \neq \emptyset)$ s. t. both X_1 and X_2 are sets of indifference.
    * and X is not a set of indifference (one set is not enough).
  - m-symmetric fuzzy measure μ iff:
    * there exists a partition of the universal set $\{X_1, \ldots, X_m\}$, with $X_1, \ldots, X_m \neq \emptyset$ s. t. $X_1, \ldots X_m$ are sets of indifference (and there is no partition with $m-1$ sets of indifference).

m-symmetric fuzzy measures: properties

- Properties (Miranda et al., 2002):
  - 2-symmetric fuzzy measure: $\mu(A) = M(|A \cap X_1|, |A \cap X_2|)$

2					1
1			$\mu(\{x_{1a}, x_{1b}\}, \{x_{2c}\})$		
0	0				
cardinality	0	1	2	3	4

  - μ, a m-symmetric measure w.r.t. the partition $\{X_1, \ldots, X_m\}$. Then, the number of values needed to determine μ is:

$$\big[(|X_1| + 1) \cdots (|X_m| + 1)\big] - 2$$

    $\rightarrow$ μ can be represented by a $(|X_1| + 1) \cdots (|X_m| + 1)$ matrix M
  - all fuzzy measures are at most N-symmetric f. m. for $N = |X|$.

Example: 2-symmetric

- Running example: it is a 2-symmetric f.m.

$$\mu(A) = M(|A \cap \{L\}|, |A \cap \{M, P\}|)$$

  - with partition (2 sets of indifference)
    1. Literary subject $\{L\}$
    2. Science subjects $\{M, P\}$,
  - probabilities only depend on cardinality (they can be ignored)
    1. $P_1(\{L\}) = 1$
    2. $P_2(\{M\}) = P_2(\{P\}) = 0.5$,
  - and distortion function f defined by

1	{L}	0.3	0.9	1.0
0	$\emptyset$	0	0.45	0.5
cardinality	sets	$\emptyset$	{M}, {P}	{M,P}
M	cardinality	0	1	2

Example: 2-dimensional distorted probabilities

- Running example: a variation that is a two dimensional d. p.

$$\mu(A) = f(P_1(A \cap \{L\}), P_2(A \cap \{M, P\}))$$

  - with partition on $X = \{M, L, P\}$
    1. Literary subject $\{L\}$
    2. Science subjects $\{M, P\}$,
  - probabilities
    1. $P_1(\{L\}) = 1$
    2. $P_2(\{M\}) = 2/3$, $P_2(\{P\}) = 1/3$,
  - and distortion function f defined by

1	{L}	0.3	0.9	0.95	1.0
0	∅	0	0.45	0.5	0.5
probability	sets	∅	{P}	{M}	{M,P}
f	probability	∅	1/3	2/3	1

m-symmetric and m-dimensional DP

- Properties:
  - Symmetric fuzzy measures are a special case of distorted probabilities:
  - All m-symmetric fuzzy measures are m-dimensional distorted probabilities.

  - If μ is a DP with $P(A) = |A|/|X|$; then, μ is a 1-symmetric f.m.
  - If μ is an m-dimensional DP with $p_i(x_j) = p_i(x_k)$ for all $x_j, x_k \in X_i$ and for all $i = 1, \ldots, m$; then, μ is a m-symmetric f.m.

OWAs

- OWA extensions:
  - OWA is equivalent to
    a Choquet integral w.r.t. to a symmetric f. m.
  - m-dimensional OWA is defined by
    the Choquet integral w.r.t. to a m-symmetric f. m.

WOWAs

- WOWA extensions:
  - WOWA is equivalent to
    a Choquet integral w.r.t. to a distorted probability
  - m-dimensional WOWA is defined by
    the Choquet integral w.r.t. to a m-dimensional distorted probability

OWAs and WOWAs

- Properties:
  - OWA is a particular case of WOWA

- m-dimensional OWA, a particular case of m-dimensional WOWA
  (it follows from: Choquet integrals w.r.t. m-symmetric fuzzy measures, particular cases of Choquet integrals w.r.t. m-dimensional distorted probabilities

Outline

- Introduction
- Distorted Probabilities
- Extension of Distorted Probabilities
- m-dimensional Distorted Probabilities
- Distorted Probabilities and OWA
- Conclusions and future work

Conclusions and Summary

- Review of distorted probabilities
- and of m-dimensional distorted probabilities
- and a few relationships with other fuzzy measures

Future work

- Counting the number of 2-dimensional distorted probabilities.
  - Assume $X = \{1, 2, 3\}$.
  - Assume μ with $\mu(\{1\}) < \mu(\{2\}) < \mu(\{3\})$.
  - Possible 2-dimensional distorted probabilities:

$\emptyset < \mu(\{1\}) < \mu(\{2\}) < \mu(\{3\}) < \mu(\{1,2\}) < \mu(\{1,3\}) < \mu(\{2,3\}) < \mu(X)$
$\emptyset < \mu(\{1\}) < \mu(\{2\}) < \mu(\{1,2\}) < \mu(\{3\}) < \mu(\{1,3\}) < \mu(\{2,3\}) < \mu(X)$
With $X_2 = \{1,2\} \rightarrow (1 < 2 => 13 < 23)$
$\emptyset < \mu(\{1\}) < \mu(\{2\}) < \mu(\{3\}) < \mu(\{1,3\}) < \mu(\{1,2\}) < \mu(\{2,3\}) < \mu(X)$
$\emptyset < \mu(\{1\}) < \mu(\{2\}) < \mu(\{3\}) < \mu(\{1,3\}) < \mu(\{2,3\}) < \mu(\{1,2\}) < \mu(X)$
With $X_2 = \{1,3\} \rightarrow (1 < 3 => 12 < 23)$
$\emptyset < \mu(\{1\}) < \mu(\{2\}) < \mu(\{1,2\}) < \mu(\{3\}) < \mu(\{2,3\}) < \mu(\{1,3\}) < \mu(X)$
$\emptyset < \mu(\{1\}) < \mu(\{2\}) < \mu(\{3\}) < \mu(\{1,2\}) < \mu(\{2,3\}) < \mu(\{1,3\}) < \mu(X)$
With $X_2 = \{2,3\} \rightarrow (2 < 3 => 12 < 13)$
$\emptyset < \mu(\{1\}) < \mu(\{2\}) < \mu(\{3\}) < \mu(\{2,3\}) < \mu(\{1,2\}) < \mu(\{1,3\}) < \mu(X)$
Invalid case:
$\emptyset < \mu(\{1\}) < \mu(\{2\}) < \mu(\{3\}) < \mu(\{2,3\}) < \mu(\{1,3\}) < \mu(\{1,2\}) < \mu(X)$

Future work

- Learning fuzzy measures from examples (case with Choquet integral)
  - Simple for k-additive f.m., for a given k
    $\longrightarrow$ optimal solution can be found
  - Relatively simple for distorted probabilities
    $\longrightarrow$ no optimal solution
  - Not easy for m-dimensional distorted probabilities

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Thank-you