Aggregation functions and information fusion. Modeling decisions

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Summary

- Aggregation functions
- There is life beyond the (weighted) mean
- Important concepts: the Pareto front (what is relevant to study)
- Fuzzy integrals to express non-independence
- Indices and methods to select functions and their parameters

Index (I)

- I. An introduction
  - 1. Defining the problem
  - 2. The goals of the field (aggregation functions)
- II. On the definition of aggregation functions
  - 1. Definition from properties
  - 2. Definition heuristically
  - 3. Definition from examples

Index (II)

- III. From the weighted mean to fuzzy integrals
  - 1. An example
  - 2. WM, OWA; and WOWA operators
  - 3. Choquet integral
  - 4. Weighted minimum and maximum
  - 5. Sugeno integral
- IV. Fuzzy measures
- V. Preference relations
- VI. Related topics
- VII. Summary

Aggregation:

I. an introduction

Aggregation:

I. an introduction

1. defining the problem

Aggregation functions

- Aggregation and information fusion
  - How to combine information
  - Focus here: information about criteria to make decisions
- In general,
  - it is a broad area, with different types of applications
    fusion for robotics (sensors), computer vision, running times, economy (GDP), biology (DNA sequences), education, ...

Aggregation functions

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  (aggregate, combine, fuse information)

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  - Well known examples for numerical data:
    - $\star$ $\sum_{i=1}^N a_i/N$ (AM arithmetic mean)
    - $\star$ $\sum_{i=1}^N p_i \cdot a_i$ (WM weighted mean, p weights)

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- Why study them, why different functions?
  - Different functions, lead to different results
    In decision, different orderings, different selections!
  - Some functions lead to inconsistent results
    Expected properties on the aggregated data!

Aggregation functions

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- Example (book II). 0 (min) – 100 (max)

	Number of seats	Security	Price	Confort	trunk
Ford T	0	20	0	20	0
Seat 600	60	0	100	0	50
Simca 1000	100	30	100	50	70
VW Beetle	80	50	30	70	100
CitroÃ<<n Acadiane	20	40	60	40	0

  - WM with $p_{\#Seats} = 0.5, p_{price} = 0.5$: Select Simca 1000
  - WM with $p_{Security} = 0.5, p_{Confort} = 0.5$: Select VW Beetle

Aggregation functions

- Aggregation functions: Other examples (p. 92, book)
  - $\log((\sum e^{a_i})/N)$ (EM, exponential mean)
  - $(\log_c((1/N)\sum c^{1/a_i}))^{-1}$ (Radical mean)

- Some functions lead to inconsistent results: *Expected* properties

- Example 4.21 (book). Assessing performance of Java runtime systems.
  - 7 benchmark programs. How aggregation should be done?
    Can we use our *nice* means: exponential mean ?, radical mean?
  - NO!!: If we want to have consistent results when time is changed from seconds to miliseconds, or to minutes!

Runtime system	227-mtr	202-jess	201-compress	209-db	222-mpegaudio	228-jack	213-javac	GM
Sun JDK 1.5.0 Client VM	325	221	204	43.4	251	192	96.1	162.13
Sun JDK 1.4.2 Client VM	318	186	199	43.6	249	181	90.9	154.51
Kaffe	32	21.3	191	24.8	101	32.9	21.3	41.95

Aggregation functions

- Aggregation functions and information fusion (in general, not restricted to decision): (Book, Ch. 1)
  - used to produce a comprehensive and specific datum about an entity,
  - datum produced from information supplied by different information sources (or the same source over time).
  - Aggregation to reduce noise, increase precision, summarize information, extract information, make decisions, etc.

Aggregation functions

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  - Information integration
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    concrete process to combine several data into a single datum.
  - Aggregation functions: $\mathbb{C} : D^N \to D$ ($\mathbb{C}$ from $\mathbb{C}$onsensus)
    $\to \mathbb{C}$ with parameters (background knowledge): $\mathbb{C}_P$

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  - Symmetry: For all permutation π over $\{1, \ldots, N\}$
    $\mathbb{C}(a_1, \ldots, a_N) = \mathbb{C}(a_{\pi(1)}, \ldots, a_{\pi(N)})$

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    $\mathbb{C}(a_1, \ldots, a_N) = \mathbb{C}(a_{\pi(1)}, \ldots, a_{\pi(N)})$
  - Unanimity + monotonicity $\to$ internality:
    $\min_i a_i \leq \mathbb{C}(a_1, \ldots, a_N) \leq \max_i a_i$

Aggregation functions

- Unanimity and idempotency:
  $\mathbb{C}(a, \ldots, a) = a$ for $a = 0$ and $a = 1$.
  - t-norms: associative, symmetric, monotonic, and has a neutral element 1
  - t-conorms: associative, symmetric, monotonic and has a neutral element 0
  - uninorms: associative, symmetric, monotonic and has a neutral element e in $(0, 1)$

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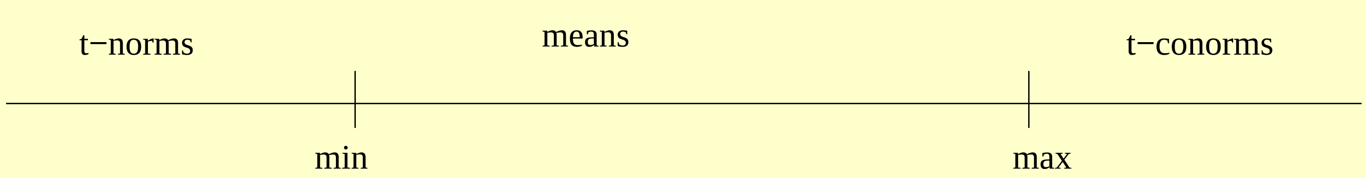

Aggregation:

I. an introduction

2. the goals of the field

Aggregation functions

- Goals of data aggregation (*goals of the area*):

Aggregation functions

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  - Formalization of the aggregation process
    - Definition of new functions (and impossibility results!)
    - Selection of functions

      (methods to decide which is the most appropriate function in a given context)
    - Parameter determination

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  - Formalization of the aggregation process
    - Definition of new functions (and impossibility results!)
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      (methods to decide which is the most appropriate function in a given context)
    - Parameter determination
  - Study of existing methods:
    - Caracterization of functions
    - Determination of the modeling capabilities of the functions
    - Relation between operators and parameters

      (how parameters influence the result: dictatorship?, sensitivity to data $\rightarrow$ index).

Aggregation:

II. on the definition of aggregation functions

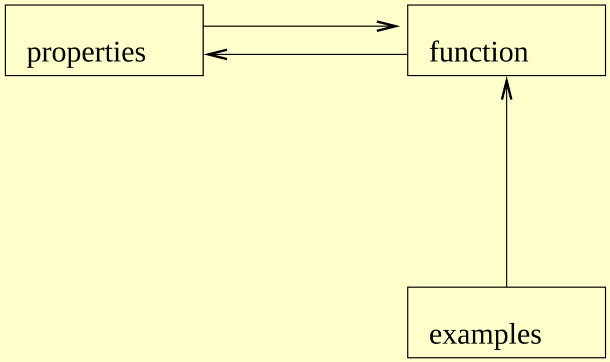
Aggregation functions

Definition of aggregation functions:

- 1. Definition from properties:
  E.g., results consistent with changes of scale:

$$\mathbb{C}(ra_1, \ldots, ra_n) = r\mathbb{C}(a_1, \ldots, a_n)$$

- 2. Heuristic definition: from functions to properties
  E.g., after some testing we decide to use $xxxxx$: properties?
- 3. Definition from examples
  E.g., find $\mathbb{C}$ that approximates the examples



Aggregation functions

- 1. Definition from properties

$$\boxed{\text{properties}} \longrightarrow \boxed{\text{function}}$$

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- Some alternatives

  (1.a) Expressing properties as equations: functional equations

Aggregation functions

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properties $\longrightarrow$ function

- Some alternatives

(1.a) Expressing properties as equations: functional equations
(1.b) Aggregation of a_1, a_2, ..., $a_N \in D$, as the datum c which is at a minimum distance from a_i:

$$\mathbb{C}(a_1, a_2, \ldots, a_N) = \arg\min_c \{\sum_{a_i} d(c, a_i)\},$$

d is a distance over D.

Aggregation functions

- Functional equations (case 1.a).
  - What is a functional equation ?
    - Equations where the unknown are functions
  - Example. Cauchy equation (a well known functional equation):

$$\phi(x + y) = \phi(x) + \phi(y)$$

  - find ϕ !

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  - Example. Cauchy equation (a well known functional equation):

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  - find ϕ !
  - Solution: For continuous ϕ,
    $\phi(x) = \alpha x$ for an arbitrary value for α

Aggregation functions

- Functional equations (case 1.a). Example I in aggregation.
  - Distribute s euros among m projects according to the opinion of N experts

	Proj 1	Proj 2	$\cdots$	Proj j	$\cdots$	Proj m
E_1	x_1^1	x_2^1	$\cdots$	x_j^1	$\cdots$	x_m^1
E_2	x_1^2	x_2^2	$\cdots$	x_j^2	$\cdots$	x_m^2
	$\vdots$	$\vdots$		$\vdots$		$\vdots$
E_i	x_1^i	x_2^i	$\cdots$	x_j^i	$\cdots$	x_m^i
	$\vdots$	$\vdots$		$\vdots$		$\vdots$
E_N	x_1^N	x_2^N	$\cdots$	x_j^N	$\cdots$	x_m^N
DM	$f_1(\mathbf{x_1})$	$f_2(\mathbf{x_2})$	$\cdots$	$f_j(\mathbf{x_j})$	$\cdots$	$f_m(\mathbf{x_m})$

Aggregation functions

- Functional equations (case 1.a). Example I in aggregation.
  - The general solution of the system (Prop. 3.11) for $m > 2$

$$f_j : [0, s]^N \to \mathbb{R}^+ \text{ for } j = \{1, \cdots, m\}$$

$$\sum_{j=1}^{m} \mathbf{x}_j = \mathbf{s} \text{ implies that } \sum_{j=1}^{m} f_j(\mathbf{x}_j) = s$$

$$f_j(\mathbf{0}) = 0 \text{ for } j = 1, \cdots, m$$

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is given by

$$f_1(\mathbf{x}) = f_2(\mathbf{x}) = \cdots = f_m(\mathbf{x}) = f((x_1, x_2, \ldots, x_N)) = \sum_{i=1}^{N} \alpha_i x_i,$$

where $\alpha_1, \cdots, \alpha_N$ are nonnegative constants satisfying $\sum_{i=1}^{N} \alpha_i = 1$, but are otherwise arbitrary.

Aggregation functions

- Functional equations (case 1.a). Example II in aggregation.
  - (Prop. 4.17) An operator $\mathbb{C}$ is separable in terms of a unique monotone increasing g (that is, $\mathbb{C}(a_1, \ldots, a_N) = g(a_1) \circ \cdots \circ g(a_N)$) (with $\circ$ continuous, associative, and cancellative)

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$$\mathbb{C}(a_1, \ldots, a_N) = \phi^{-1}(\sum_{i=1}^{N} \phi(g(a_i)))$$

and satisfies unanimity

$$\mathbb{C}(a, \ldots, a) = a$$

if and only if it is of the form (quasi-arithmetic mean)

$$\mathbb{C}(a_1, \ldots, a_N) = \phi^{-1}(\frac{1}{N} \sum_{i=1}^{N} \phi(a_i)).$$

Aggregation functions

- Functional equations (case 1.a). Quasi-arithmetic means

Name	Generator function	$\mathbb{C}(a_1, \ldots, a_N)$
Arithmetic mean	$\phi(x) = x$	$\frac{\sum_{i=1}^N x_i}{N}$
Geometric mean	$\phi(x) = \log x$	$\sqrt[N]{\frac{\sum_{i=1}^N x_i}{N}}$
Harmonic mean	$\phi(x) = 1/x$	$\frac{N}{\sum_{i=1}^N \frac{1}{x_i}}$
Root-mean-square	$\phi(x) = x^2$	$\sqrt{\frac{\sum_{i=1}^N x_i^2}{N}}$
Root-mean-power	$\phi(x) = x^\alpha$	$\sqrt[\alpha]{\frac{\sum_{i=1}^N x_i^\alpha}{N}}$
Exponential mean	$\phi(x) = e^x$	$\log\left(\frac{\sum_{i=1}^N e^{x_i}}{N}\right)$
Radical mean	$\phi(x) = c^{1/x}$	$\left(\log_c\left(\frac{\sum_{i=1}^N c^{1/x_i}}{N}\right)\right)^{-1}$
Basis-exponential mean	$\phi(x) = x^x$	m s.t. $m^m = \frac{\sum_{i=1}^N x_i^{x_i}}{N}$
Basis-radical mean	$\phi(x) = x^{1/x}$	m s.t. $m^{1/m} = \frac{\sum_{i=1}^N x_i^{1/x_i}}{N}$

Aggregation functions

- Functional equations (case 1.a). Example III in aggregation.
  - (Prop. 4.20) An operator $\mathbb{C}$ is separable in terms of a unique monotone increasing g (that is, $\mathbb{C}(a_1, \ldots, a_N) = g(a_1) \circ \cdots \circ g(a_N)$) (with $\circ$ continuous, associative, and cancellative)

$$\mathbb{C}(a_1, \ldots, a_N) = \phi^{-1}(\sum_{i=1}^{N} \phi(g(a_i)))$$

and satisfies unanimity

$$\mathbb{C}(a, \ldots, a) = a$$

and positive homogeneity

$$\mathbb{C}(ra_1, \ldots, ra_N) = r\mathbb{C}(a_1, \ldots, a_N)$$

Aggregation functions

- Functional equations (case 1.a). Example III in aggregation (cont.)
  - if and only if $\mathbb{C}$ is either the root-mean-power

$$\mathbb{C}(a_1, \ldots, a_N) = (\frac{1}{N}\sum_{i=1}^{N} a_i^{\alpha})^{1/\alpha}$$

with parameter $\alpha \neq 0$ (RMP_α) or the geometric mean

$$\mathbb{C}(a_1, \ldots, a_N) = (\prod_{i=1}^{N} a_i)^{1/N}.$$

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$$\mathbb{C}(a_1, \ldots, a_N) = (\prod_{i=1}^{N} a_i)^{1/N}.$$

- Remember Ex. 4.21. Assessing performance of Java runtime systems. Only RMP and GM are acceptable !! (Note: $\lim_{\alpha \to 0} RMP_\alpha = GM$)
- Root-mean-powers, known as rth power mean, generalized mean.

Aggregation functions

- Functional equations (case 1.a). Example IV in aggregation.
  - (Prop. 4.24) When we add the equation of reciprocity

$$\mathbb{C}(1/a_1, \ldots, 1/a_N) = 1/\mathbb{C}(a_1, \ldots, a_N)$$

    the only operator $\mathbb{C}$ satisfying all conditions is the geometric mean

$$\mathbb{C}(a_1, \ldots, a_N) = (\prod_{i=1}^{N} a_i)^{1/N}.$$

Aggregation functions

- Functional equations (case 1.a). Root-mean-powers (rth power mean, generalized mean)

Name	α	$\mathbb{C}(a_1, \ldots, a_N)$
Arithmetic mean	$\alpha = 1$	$\frac{\sum_{i=1}^{N} x_i}{N}$
Root-mean-square	$\alpha = 2$	$\sqrt{\frac{\sum_{i=1}^{N} x_i^2}{N}}$
Harmonic mean	$\alpha = -1$	$\frac{N}{\sum_{i=1}^{N} \frac{1}{x_i}}$

Aggregation functions

- Example (case (b)): Consider the following expression

$$\mathbb{C}(a_1, a_2, \ldots, a_N) = \arg\min_c \{ \sum_{a_i} d(c, a_i) \},$$

where a_i are numbers from $\mathbb{R}$ and d is a distance on D. Then,

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- Example (case (b)): Consider the following expression

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where a_i are numbers from $\mathbb{R}$ and d is a distance on D. Then,

1. When $d(a,b) = (a-b)^2$, $\mathbb{C}$ is the arithmetic mean
   I.e., $\mathbb{C}(a_1, a_2, \ldots, a_N) = \sum_{i=1}^{N} a_i/N$.
2. When $d(a,b) = |a-b|$, $\mathbb{C}$ is the median
   I.e., the median of $a_1, a_2, \ldots, a_N$ is the element which occupies the central position when we order a_i.
3. When $d(a,b) = 1$ iff $a = b$, $\mathbb{C}$ is the plurality rule (mode or voting).
   I.e., $\mathbb{C}(a_1, a_2, \ldots, a_N)$ selects the element of $\mathbb{R}$ with a largest frequency among elements in $(a_1, a_2, \ldots, a_N)$.

Aggregation functions

- 2. Heuristic definition: from functions to properties

  function $\longrightarrow$ properties

  - We can use functional equations for this purpose:
    characterizations of functions
  - We have propositions with "if and only if"

    eq1, eq2, eq3 if and only if aggr. function

    so, given a function we know the *fundamental properties* of the function.

    E.g., geometric mean: a fundamental property is reciprocity

- Note: characterizations are not unique

Aggregation:

III. from the weighted mean to fuzzy integrals

Aggregation:

III. from the weighted mean to fuzzy integrals

1. An example

Aggregation: example

Example. A and B teaching a tutorial+training course w/ constraints

- The total number of sessions is six.
- Professor A will give the tutorial, which should consist of about three sessions; three is the optimal number of sessions; a difference in the number of sessions greater than two is unacceptable.
- Professor B will give the training part, consisting of about two sessions.
- Both professors should give more or less the same number of sessions. A difference of one or two is half acceptable; a difference of three is unacceptable.

Aggregation: example

Example. Formalization

- Variables
  - x_A: Number of sessions taught by Professor A
  - x_B: Number of sessions taught by Professor B
- Constraints
  - the constraints are translated into
    - ⋆ C_1: $x_A + x_B$ should be about 6
    - ⋆ C_2: x_A should be about 3
    - ⋆ C_3: x_B should be about 2
    - ⋆ C_4: $|x_A - x_B|$ should be about 0
  - using fuzzy sets, the constraints are described ...

Aggregation: example

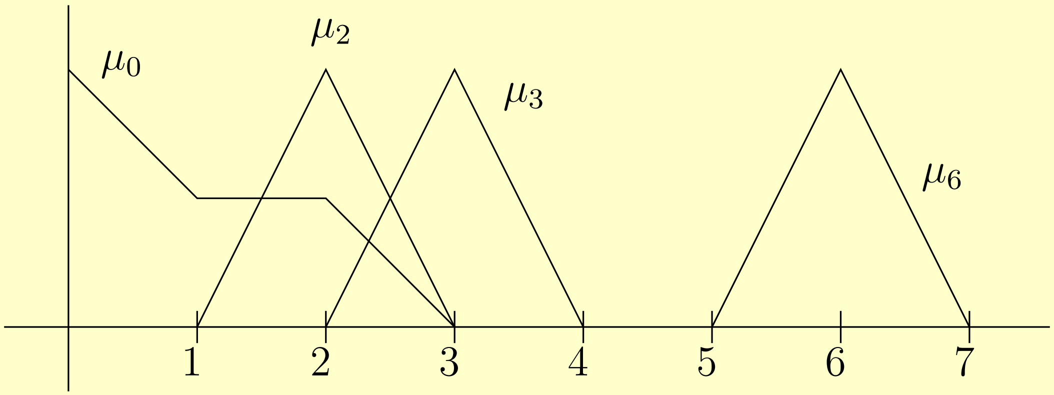
Example. Formalization

- Constraints
  - if fuzzy set μ_6 expresses "about 6," then,
    we evaluate "$x_A + x_B$ should be about 6" by $\mu_6(x_A + x_B)$.
    $\rightarrow$ given μ_6, μ_3, μ_2, μ_0,
  - Then, given a solution pair (x_A, x_B), the degrees of satisfaction:
    - ⋆ $\mu_6(x_A + x_B)$
    - ⋆ $\mu_3(x_A)$
    - ⋆ $\mu_2(x_B)$
    - ⋆ $\mu_0(|x_A - x_B|)$

Aggregation: example

Example. Formalization

- Membership functions for constraints



Aggregation: example

Example. Application

alternative	Satisfaction degrees	Satisfaction degrees			
(x_A, x_B)	$(\mu_6(x_A + x_B),\ \mu_3(x_A),\ \mu_2(x_B),\ \mu_0(\lvert x_A - x_B\rvert))$	C_1	C_2	C_3	C_4
$(2, 2)$	$(\mu_6(4),\ \mu_3(2),\ \mu_2(2),\ \mu_0(0))$	0	0.5	1	1
$(2, 3)$	$(\mu_6(5),\ \mu_3(2),\ \mu_2(3),\ \mu_0(1))$	0.5	0.5	0.5	0.5
$(2, 4)$	$(\mu_6(6),\ \mu_3(2),\ \mu_2(4),\ \mu_0(2))$	1	0.5	0	0.5
$(3.5, 2.5)$	$(\mu_6(6),\ \mu_3(3.5),\ \mu_2(2.5),\ \mu_0(1))$	1	0.5	0.5	0.5
$(3, 2)$	$(\mu_6(5),\ \mu_3(3),\ \mu_2(2),\ \mu_0(1))$	0.5	1	1	0.5
$(3, 3)$	$(\mu_6(6),\ \mu_3(3),\ \mu_2(3),\ \mu_0(0))$	1	1	0.5	1

Aggregation:

III. from the weighted mean to fuzzy integrals

2. WM, OWA, and WOWA operators

Aggregation: WM, OWA, and WOWA operators

- Operators
  - Weighting vector (dimension N): $v = (v_1 ... v_N)$ iff $v_i \in [0,1]$ and $\sum_i v_i = 1$
  - Arithmetic mean (AM :$\mathbb{R}^N \to \mathbb{R}$): $AM(a_1, ..., a_N) = (1/N) \sum_{i=1}^N a_i$
  - Weighted mean (WM: $\mathbb{R}^N \to \mathbb{R}$): $WM_{\mathbf{p}}(a_1, ..., a_N) = \sum_{i=1}^N p_i a_i$ ($\mathbf{p}$ a weighting vector of dimension N)
  - Ordered Weighting Averaging operator (OWA: $\mathbb{R}^N \to \mathbb{R}$):

$$OWA_{\mathbf{w}}(a_1, ..., a_N) = \sum_{i=1}^{N} w_i a_{\sigma(i)},$$

where $\{\sigma(1), ..., \sigma(N)\}$ is a permutation of $\{1, ..., N\}$ s. t. $a_{\sigma(i-1)} \geq a_{\sigma(i)}$, and $\mathbf{w}$ a weighting vector.

Aggregation: WM, OWA, and WOWA operators

Example. Application

- Let us consider the following situation:
  - Professor A is more important than Professor B
  - The number of sessions equal to six is the most important constraint (not a *crisp* requirement)
  - The difference in the number of sessions taught by the two professors is the least important constraint

WM with $\mathbf{p} = (p_1, p_2, p_3, p_4) = (0.5, 0.3, 0.15, 0.05)$.

Aggregation: WM, OWA, and WOWA operators

Example. Application

- WM with $\mathbf{p} = (p_1, p_2, p_3, p_4) = (0.5, 0.3, 0.15, 0.05)$.

alternative	Aggregation of the Satisfaction degrees	WM
(x_A, x_B)	$WM_{\mathbf{p}}(C_1, C_2, C_3, C_4)$	
$(2, 2)$	$WM_{\mathbf{p}}(0, 0.5, 1, 1)$	0.35
$(2, 3)$	$WM_{\mathbf{p}}(0.5, 0.5, 0.5, 0.5)$	0.5
$(2, 4)$	$WM_{\mathbf{p}}(1, 0.5, 0, 0.5)$	0.675
$(3.5, 2.5)$	$WM_{\mathbf{p}}(1, 0.5, 0.5, 0.5)$	0.75
$(3, 2)$	$WM_{\mathbf{p}}(0.5, 1, 1, 0.5)$	0.725
$(3, 3)$	$WM_{\mathbf{p}}(1, 1, 0.5, 1)$	0.925

Aggregation: WM, OWA, and WOWA operators

Example. Application

- Compensation: how many values can have a bad evaluation
- One bad value does not matter: OWA with $\mathbf{w} = (1/3, 1/3, 1/3, 0)$ (lowest value discarded)

alternative	Aggregation of the Satisfaction degrees	OWA
(x_A, x_B)	$OWA_{\mathbf{w}}(C_1, C_2, C_3, C_4)$	
$(2, 2)$	$OWA_{\mathbf{w}}(0, 0.5, 1, 1)$	0.8333
$(2, 3)$	$OWA_{\mathbf{w}}(0.5, 0.5, 0.5, 0.5)$	0.5
$(2, 4)$	$OWA_{\mathbf{w}}(1, 0.5, 0, 0.5)$	0.6666
$(3.5, 2.5)$	$OWA_{\mathbf{w}}(1, 0.5, 0.5, 0.5)$	0.6666
$(3, 2)$	$OWA_{\mathbf{w}}(0.5, 1, 1, 0.5)$	0.8333
$(3, 3)$	$OWA_{\mathbf{w}}(1, 1, 0.5, 1)$	1.0

Aggregation: WM, OWA, and WOWA operators

- Weighted Ordered Weighted Averaging WOWA operator (WOWA $:\mathbb{R}^N \to \mathbb{R}$):

$$WOWA_{\mathbf{p},\mathbf{w}}(a_1, ..., a_N) = \sum_{i=1}^{N} \omega_i a_{\sigma(i)}$$

where

$$\omega_i = w^*(\sum_{j \leq i} p_{\sigma(j)}) - w^*(\sum_{j < i} p_{\sigma(j)}),$$

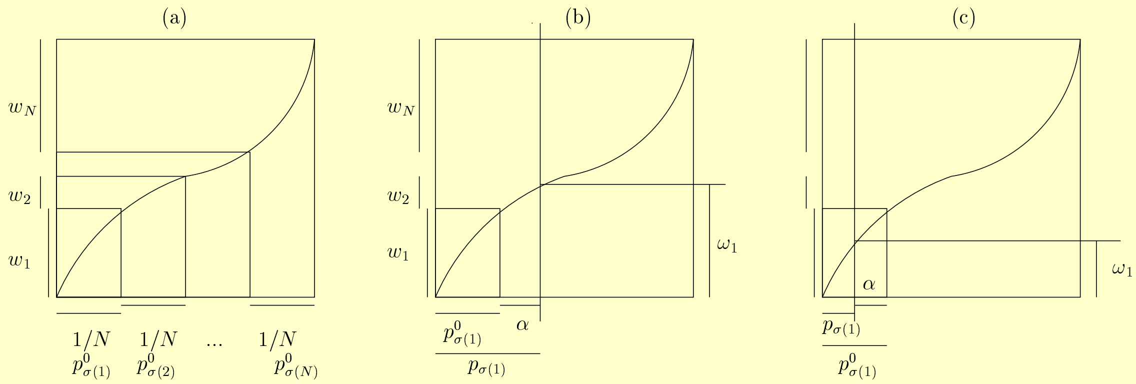
with σ a permutation of $\{1, ..., N\}$ s. t. $a_{\sigma(i-1)} \geq a_{\sigma(i)}$, and w^* a nondecreasing function that interpolates the points

$$\{(i/N, \sum_{j \leq i} w_j)\}_{i=1,...,N} \cup \{(0,0)\}.$$

w^* is required to be a straight line when the points can be interpolated in this way.

Aggregation: WM, OWA, and WOWA operators

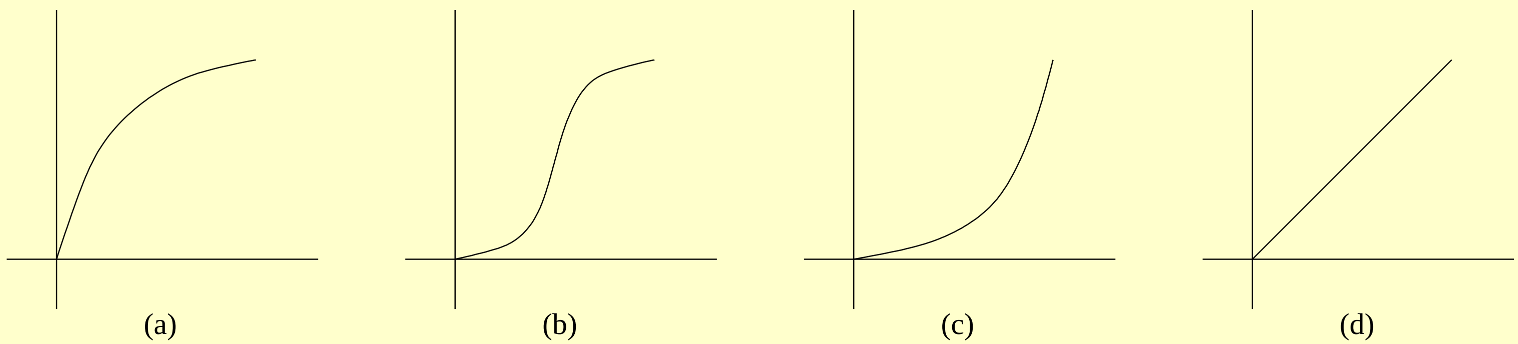
- Construction of the w^* quantifier



- Rationale for new weights (ω_i, for each value a_i) in terms of $\mathbf{p}$ and $\mathbf{w}$.
  - If a_i is small, and small values have more importance than larger ones, increase p_i for a_i (i.e., $\omega_i \geq p_{\sigma(i)}$).
    (the same holds if the value a_i is large and importance is given to large values)
  - If a_i is small, and importance is for large values, $\omega_i < p_{\sigma(i)}$
    (the same holds if a_i is large and importance is given to small values).

Aggregation: WM, OWA, and WOWA operators

- The shape of the function w^* gives importance
  - (a) to large values
  - (b) to medium values
  - (c) to small values
  - (d) equal importance to all values



Aggregation: WM, OWA, and WOWA operators

Example. Application

- Importance for constraints as given above: $\mathbf{p} = (0.5, 0.3, 0.15, 0.05)$
- Compensation as given above: $\mathbf{w} = (1/3, 1/3, 1/3, 0)$ (lowest value discarded)

  $\rightarrow$ WOWA with $\mathbf{p}$ and $\mathbf{w}$.

alternative	Aggregation of the Satisfaction degrees	WOWA
(x_A, x_B)	$WOWA_{\mathbf{p},\mathbf{w}}(C_1, C_2, C_3, C_4)$	
$(2, 2)$	$WOWA_{\mathbf{p},\mathbf{w}}(0, 0.5, 1, 1)$	0.4666
$(2, 3)$	$WOWA_{\mathbf{p},\mathbf{w}}(0.5, 0.5, 0.5, 0.5)$	0.5
$(2, 4)$	$WOWA_{\mathbf{p},\mathbf{w}}(1, 0.5, 0, 0.5)$	0.8333
$(3.5, 2.5)$	$WOWA_{\mathbf{p},\mathbf{w}}(1, 0.5, 0.5, 0.5)$	0.8333
$(3, 2)$	$WOWA_{\mathbf{p},\mathbf{w}}(0.5, 1, 1, 0.5)$	0.8
$(3, 3)$	$WOWA_{\mathbf{p},\mathbf{w}}(1, 1, 0.5, 1)$	1.0

Aggregation: WM, OWA, and WOWA operators

- Properties
  - The WOWA operator generalizes the WM and the OWA operator.
    - When $\mathbf{p} = (1/N \; \ldots \; 1/N)$, OWA

$$WOWA_{\mathbf{p},\mathbf{w}}(a_1, ..., a_N) = OWA_{\mathbf{w}}(a_1, ..., a_N) \text{ for all } \mathbf{w} \text{ and } a_i.$$

    - When $\mathbf{w} = (1/N \ldots 1/N)$, WM

$$WOWA_{\mathbf{p},\mathbf{w}}(a_1, ..., a_N) = WM_{\mathbf{p}}(a_1, ..., a_N) \text{ for all } \mathbf{p} \text{ and } a_i.$$

    - When $\mathbf{w} = \mathbf{p} = (1/N \ldots 1/N)$, AM

$$WOWA_{\mathbf{p},\mathbf{w}}(a_1, ..., a_N) = AM(a_1, ..., a_N)$$

Aggregation:

III. from the weighted mean to fuzzy integrals

3. Choquet integral

Choquet integrals

- In the WM, a single weight is used for each element
  I.e., $p_i = p(x_i)$ (where, x_i is the information source that supplies a_i)
  $\rightarrow$ when we consider a set $A \subset X$, *weight* of$A$???

Choquet integrals

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  I.e., $p_i = p(x_i)$ (where, x_i is the information source that supplies a_i)
  $\rightarrow$ when we consider a set $A \subset X$, *weight* of$A$???

... fuzzy measures $\mu(A)$

Choquet integrals

- Example.
  - We need to evaluate students (who is best?) using marks in three subjects
    X={*Mathematics, Physics, Literature*} (M,P,L)
  - $p_M = 0.4$, $p_P = 0.4$, $p_L = 0.2$.
    In the WM, a single weight is used for each element
    I.e., $p_i = p(x_i)$ (where, x_i is the information source that supplies a_i)
    $\rightarrow$ when we consider a set $A \subset X$, *weight* of$A$???
  - *p(Mathematics, Physics) ?*

Choquet integrals

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  - We need to evaluate students (who is best?) using marks in three subjects
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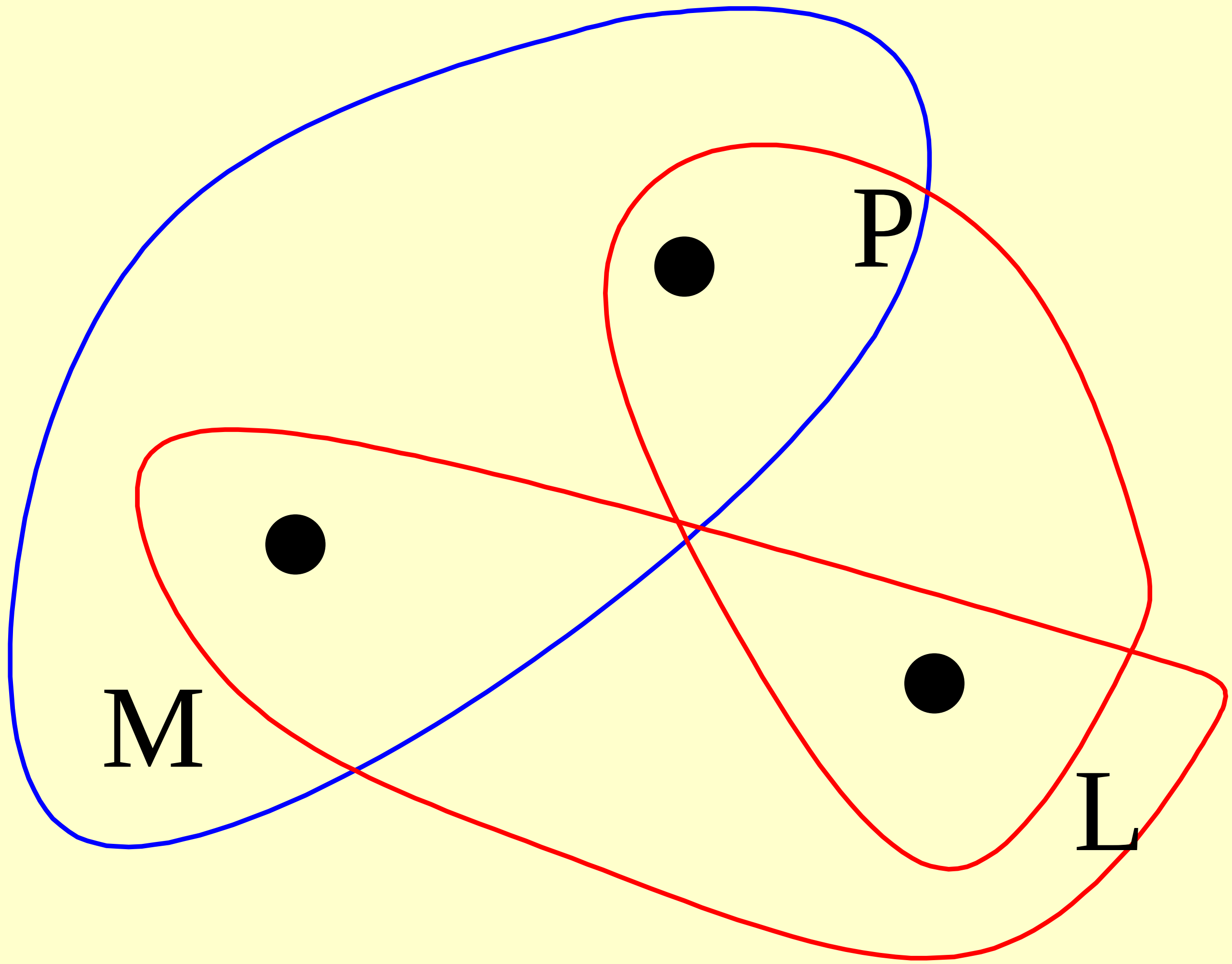
    In the WM, a single weight is used for each element
    I.e., $p_i = p(x_i)$ (where, x_i is the information source that supplies a_i)
    $\rightarrow$ when we consider a set $A \subset X$, *weight* of A???
  - *p(Mathematics, Physics) ?*

. . . fuzzy measures $\mu(A)$

Choquet integrals

- fuzzy measures $\mu(A)$: $X \subset X = \{M, P, L\}$



Choquet integrals

- fuzzy measures $\mu(A)$
  Formally,
  - Fuzzy measure ($\mu : \wp(X) \rightarrow [0, 1]$), a set function satisfying
  (i) $\mu(\emptyset) = 0$, $\mu(X) = 1$ (boundary conditions)
  (ii) $A \subseteq B$ implies $\mu(A) \leq \mu(B)$ (monotonicity)

Choquet integrals

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  Formally,

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  ◦ Similar to probability or standard (additive) measures,
  but additivity condition is removed
  replaced by monotonicity

Choquet integrals

- fuzzy measures $\mu(A)$
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  ◦ Fuzzy measure ($\mu : \wp(X) \to [0,1]$), a set function satisfying
  (i) $\mu(\emptyset) = 0$, $\mu(X) = 1$ (boundary conditions)
  (ii) $A \subseteq B$ implies $\mu(A) \leq \mu(B)$ (monotonicity)

  ◦ Similar to probability or standard (additive) measures,
  but additivity condition is removed
  replaced by monotonicity

  ◦ Why? to represent redundancy and support (for $A \cap B = \emptyset$)
  $\mu(A \cup B) < \mu(A) + \mu(B)$
  $\mu(A \cup B) > \mu(A) + \mu(B)$

Choquet integrals

- Fuzzy measures $\mu(A)$: Example with $X = \{M, P, L\}$

1. Boundary conditions:
   $\mu(\emptyset) = 0$, $\mu(\{M, P, L\}) = 1$
2. Relative importance of scientific versus literary subjects:
   $\mu(\{M\}) = \mu(\{P\}) = 0.45$, $\mu(\{L\}) = 0.3$
3. *Redudancy* between mathematics and physics:
   $\mu(\{M, P\}) = 0.5 < \mu(\{M\}) + \mu(\{P\})$
4. *Support* between literature and scientific subjects:
   $\mu(\{M, L\}) = \mu(\{P, L\}) = 0.9 > \mu(\{P\}) + \mu(\{L\}) = 0.45 + 0.3 = 0.75$
   $\mu(\{M, L\}) = \mu(\{P, L\}) = 0.9 > \mu(\{M\}) + \mu(\{L\}) = 0.45 + 0.3 = 0.75$

Choquet integrals

- Now, we have a fuzzy measure $\mu(A)$
  then, how aggregation proceeds?
  $\Rightarrow$ fuzzy integrals as e.g. the Choquet integral

Choquet integrals

- In WM, we combine a_i w.r.t. weights p_i.
  $\rightarrow$ a_i is the value supplied by information source x_i.
  Formally

Choquet integrals

- In WM, we combine a_i w.r.t. weights p_i.
  $\rightarrow$ a_i is the value supplied by information source x_i.
  Formally
  - $X = \{x_1, \ldots, x_N\}$ is the set of information sources
  - $f : X \rightarrow \mathbb{R}^+$ the values supplied by the sources
    $\rightarrow$ then $a_i = f(x_i)$

  Thus,

$$WM_{\mathbf{p}}(a_1, ..., a_N) = \sum_{i=1}^{N} p_i a_i = \sum_{i=1}^{N} p_i f(x_i) = WM_{\mathbf{p}}(f(x_1), ..., f(x_N))$$

Choquet integrals

- Choquet integral of f w.r.t. μ (alternative notation, $CI_\mu(a_1, \ldots, a_N)/CI_\mu(f)$)

$$(C)\int f d\mu = \sum_{i=1}^{N} [f(x_{s(i)}) - f(x_{s(i-1)})]\mu(A_{s(i)}),$$

where s in $f(x_{s(i)})$ is a permutation so that $f(x_{s(i-1)}) \leq f(x_{s(i)})$ for $i \geq 1$, $f(x_{s(0)}) = 0$, and $A_{s(k)} = \{x_{s(j)} | j \geq k\}$ and $A_{s(N+1)} = \emptyset$.

- Alternative expressions (Proposition 6.18):

$$(C)\int f d\mu = \sum_{i=1}^{N} f(x_{\sigma(i)})[\mu(A_{\sigma(i)}) - \mu(A_{\sigma(i-1)})],$$

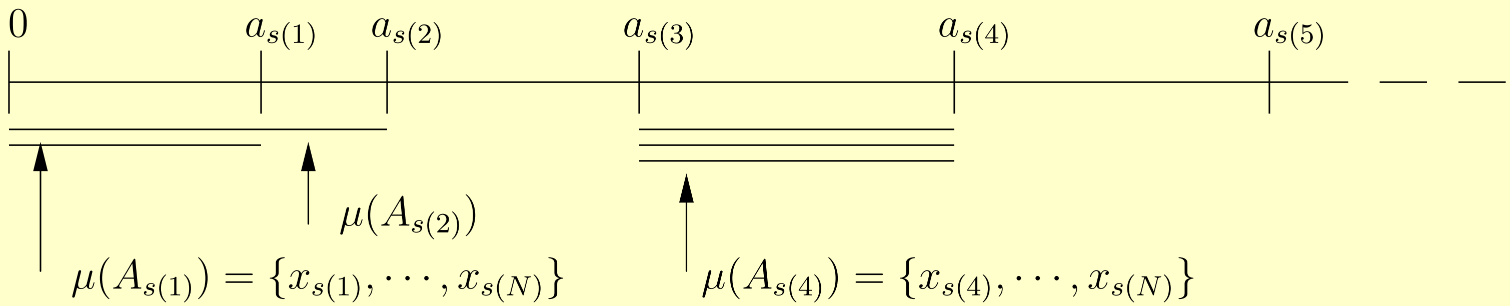
$$(C)\int f d\mu = \sum_{i=1}^{N} f(x_{s(i)})[\mu(A_{s(i)}) - \mu(A_{s(i+1)})],$$

where σ is a permutation of $\{1, \ldots, N\}$ s.t. $f(x_{\sigma(i-1)}) \geq f(x_{\sigma(i)})$, where $A_{\sigma(k)} = \{x_{\sigma(j)} | j \leq k\}$ for $k \geq 1$ and $A_{\sigma(0)} = \emptyset$

Choquet integrals

- Different equations point out different aspects of the CI

(6.1) $(C)\int f d\mu = \sum_{i=1}^{N}[f(x_{s(i)}) - f(x_{s(i-1)})]\mu(A_{s(i)})$,

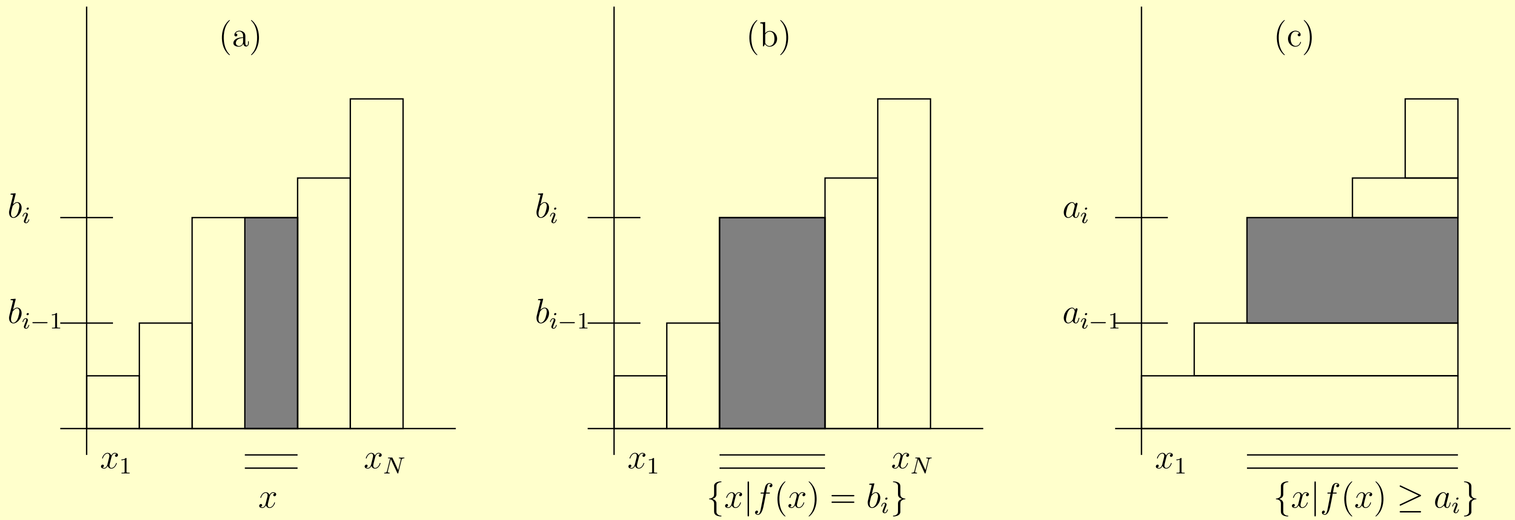


(6.2) $(C)\int f d\mu = \sum_{i=1}^{N} f(x_{\sigma(i)})[\mu(A_{\sigma(i)}) - \mu(A_{\sigma(i-1)})]$,

Choquet integrals

- $\int f d\mu =$ (for additive measures)

  (6.5) $\sum_{x \in X} f(x)\mu(\{x\})$
  (6.6) $\sum_{i=1}^{R} b_i \mu(\{x | f(x) = b_i\})$
  (6.7) $\sum_{i=1}^{N} (a_i - a_{i-1})\mu(\{x | f(x) \geq a_i\})$
  (6.8) $\sum_{i=1}^{N} (a_i - a_{i-1})\big(1 - \mu(\{x | f(x) \leq a_{i-1}\})\big)$

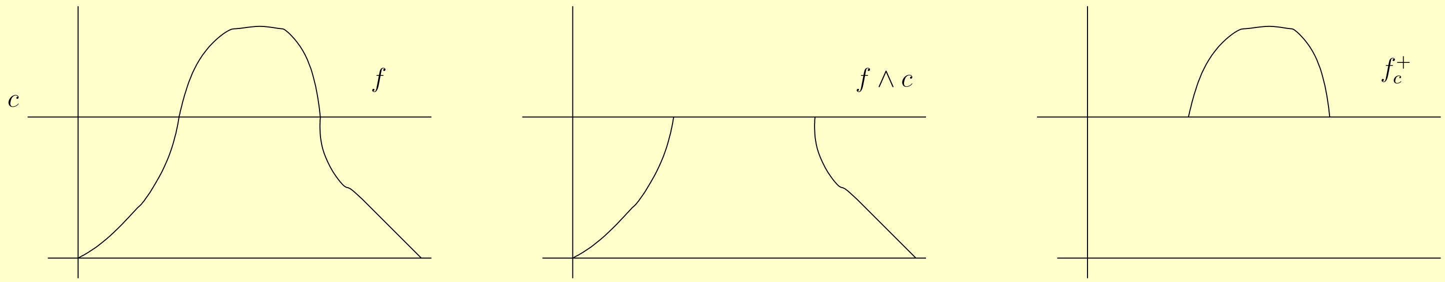


- Among (6.5), (6.6) and (6.7), only (6.7) satisfies internality.

Choquet integrals

- Properties of CI
  - Horizontal additive because $CI_\mu(f) = CI_\mu(f \wedge c) + CI_\mu(f_c^+)$
    ($f = (f \wedge c) + f_c^+$ is a horizontal additive decomposition of f)
    where, f_c^+ is defined by (for $c \in [0,1]$)

$$f_c^+ = \begin{cases} 0 & \text{if } f(x) \leq c \\ f(x) - c & \text{if } f(x) > c. \end{cases}$$



Choquet integrals

- Definitions (X a reference set, f, g functions $f, g : X \to [0, 1]$)
  - $f < g$ when, for all x_i,
  $$f(x_i) < g(x_i)$$
  - f and g are comonotonic if, for all $x_i, x_j \in X$,
  $$f(x_i) < f(x_j) \text{ imply that } g(x_i) \leq g(x_j)$$
  - $\mathbb{C}$ is comonotonic monotone if and only if, for comonotonic f and g,
  $$f \leq g \text{ imply that } \mathbb{C}(f) \leq \mathbb{C}(g)$$
  - $\mathbb{C}$ is comonotonic additive if and only if, for comonotonic f and g,
  $$\mathbb{C}(f + g) = \mathbb{C}(f) + \mathbb{C}(g)$$

- Characterization. Let $\mathbb{C}$ satisfy the following properties
  - $\mathbb{C}$ is comonotonic monotone
  - $\mathbb{C}$ is comonotonic additive
  - $\mathbb{C}(1, \ldots, 1) = 1$

  Then, there exists μ s.t. $\mathbb{C}(f)$ is the CI of f w.r.t. μ.

Choquet integrals

- Properties
  - WM, OWA and WOWA are particular cases of CI.
    - ⋆ WM with weighting vector $\mathbf{p}$ is a CI w.r.t. $\mu_{\mathbf{p}}(B) = \sum_{x_i \in B} p_i$
    - ⋆ OWA with weighting vector $\mathbf{w}$ is a CI w.r.t. $\mu_{\mathbf{w}}(B) = \sum_{i=1}^{|B|} w_i$
    - ⋆ WOWA with w.v. $\mathbf{p}$ and $\mathbf{w}$ is a CI w.r.t. $\mu_{\mathbf{p},\mathbf{w}}(B) = w^*(\sum_{x_i \in B} p_i)$
  - Any CI with a symmetric measure is an OWA operator.
  - Any CI with a distorted probability is a WOWA operator.
  - Let A be a crisp subset of X; then, the Choquet integral of A with respect to μ is $\mu(A)$.

    Here, the integral of A corresponds to the integral of its characteristic function, or, in other words, to the integral of the function f_A defined as $f_A(x) = 1$ if and only if $x \in A$.

Aggregation:

III. from the weighted mean to fuzzy integrals

4. Weighted minimum and maximum

Weighted Minimum and Weighted Maximum

- Possibilistic weighting vector (dimension N): $\mathbf{v} = (v_1 ... v_N)$ iff $v_i \in [0,1]$ and $\max_i v_i = 1$.

- Weighted minimum (WMin: $[0,1]^N \to [0,1]$):
  $WMin_{\mathbf{u}}(a_1, ..., a_N) = \min_i \max(neg(u_i), a_i)$
  (alternative definition can be given with $\mathbf{v} = (v_1, \ldots, v_N)$ where $v_i = neg(u_i)$)

- Weighted maximum (WMax: $[0,1]^N \to [0,1]$):
  $WMax_{\mathbf{u}}(a_1, ..., a_N) = \max_i \min(u_i, a_i)$

Weighted Minimum and Weighted Maximum

- Only operators in ordinal scales (max, min, neg) are used in $WMax$ and $WMin$.
- neg is completely determined in an ordinal scale

Proposition 6.36. Let $L = \{l_0, \ldots, l_r\}$ with $l_0 <_L l_1 <_L \cdots <_L l_r$; then, there exists only one function, $neg : L \to L$, satisfying

(N1) if $x <_L x'$ then $neg(x) >_L neg(x')$ for all x, x' in L.
(N2) $neg(neg(x)) = x$ for all x in L.

This function is defined by $neg(x_i) = x_{r-i} \text{ for all } x_i \text{ in } L$

- Properties. For $\mathbf{u} = (1, \ldots, 1)$
  - $WMIN_{\mathbf{u}} = \min$
  - $WMAX_{\mathbf{u}} = \max$

Aggregation:

III. from the weighted mean to fuzzy integrals

5. Sugeno integral

Sugeno integral

- Sugeno integral of f w.r.t. μ (alternative notation, $SI_\mu(a_1, \ldots, a_N)/SI_\mu(f)$)

$$(S)\int f d\mu = \max_{i=1,N} \min(f(x_{s(i)}), \mu(A_{s(i)})),$$

  where s in $f(x_{s(i)})$ is a permutation so that $f(x_{s(i-1)}) \leq f(x_{s(i)})$ for $i \geq 2$, and $A_{s(k)} = \{x_{s(j)} | j \geq k\}$.

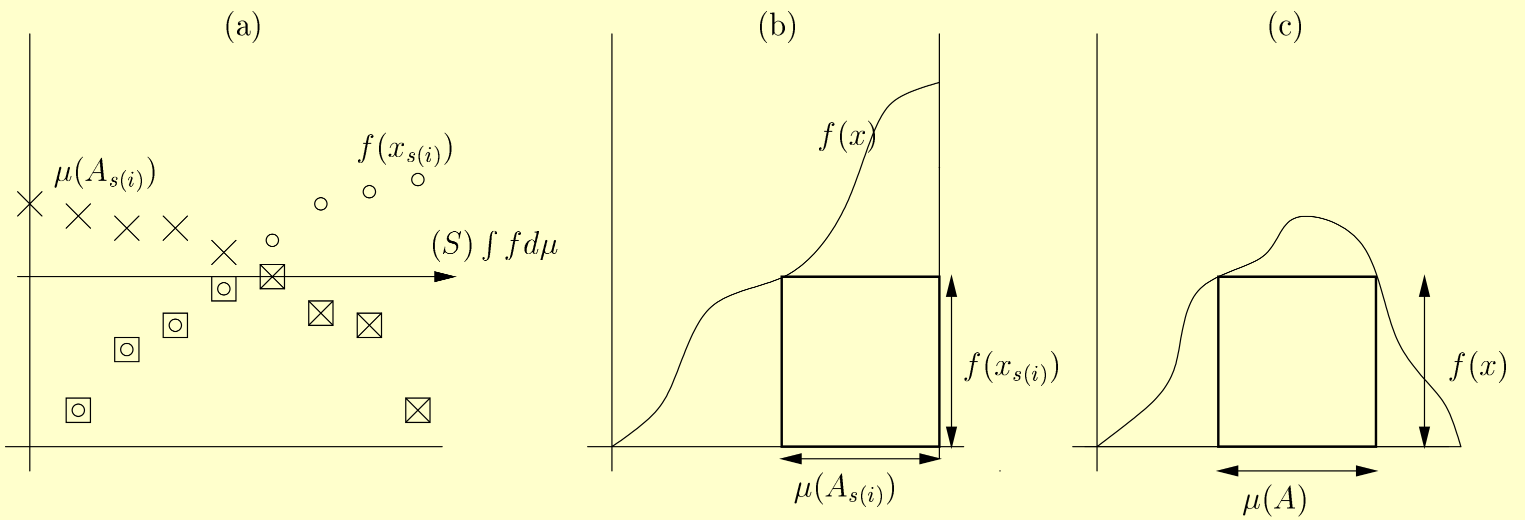
- Alternative expression (Proposition 6.38):

$$\max_i \min(f(x_{\sigma(i)}), \mu(A_{\sigma(i)})),$$

  where σ is a permutation of $\{1, \ldots, N\}$ s.t. $f(x_{\sigma(i-1)}) \geq f(x_{\sigma(i)})$, where $A_{\sigma(k)} = \{x_{\sigma(j)} | j \leq k\}$ for $k \geq 1$

Sugeno integral

- Graphical interpretation of Sugeno integrals



Sugeno integral

- Properties
  - WMin and WMax are particular cases of SI
    - ⋆ WMax with weighting vector $\mathbf{u}$ is a SI w.r.t. $\mu_{\mathbf{u}}^{wmax}(A) = \max_{a_i \in A} u_i$.
    - ⋆ WMin with weighting vector $\mathbf{u}$ is a SI w.r.t. $\mu_{\mathbf{u}}^{wmin}(A) = 1 - \max_{a_i \notin A} u_i$.

Fuzzy integrals

- Fuzzy integrals that generalize Choquet and Sugeno integrals
  - The fuzzy t-conorm integral
  - The twofold integral

Aggregation:

IV. fuzzy measures

Fuzzy measures

- Definition:

(i) $\mu(\emptyset) = 0$, $\mu(X) = 1$ (boundary conditions)
(ii) $A \subseteq B$ implies $\mu(A) \leq \mu(B)$ (monotonicity)

- Difficulty:
  - $2^{|X|} - 2$ values (because $\mu(\emptyset) = 0$, $\mu(X) = 1$)

- Solution: Families of fuzzy measures (to reduce complexity)

Fuzzy measures

- Examples of families:
  - $\perp$-Decomposable Fuzzy Measures ($\perp$ a t-conorm)

    $$\mu(A \cup B) = \mu(A) \perp \mu(B).$$

    Therefore, for a given A:

    $$\perp_{x_i \in A} v(x_i)$$

  - Sugeno λ-measures (for $\lambda > -1$)

    $$\mu(A \cup B) = \mu(A) + \mu(B) + \lambda\mu(A)\mu(B)$$

    $\perp$-decomposable for $\perp(x, y) = min(1, x + y + \lambda xy)$.
  - Extensively used in computer vision applications

Fuzzy measures

- Examples of families:
  - Distorted probabilities (P: probability, f: increasing function)

$$\mu(A) = f(P(A))$$

  - Well known in economics (decision)
  - m-dimensional distorted probability $X_1, \ldots, X_m$, P_i, f

$$\mu(A) = f(P_1(A \cap X_1), P_2(A \cap X_2), \cdots, P_m(A \cap X_m)).$$

Aggregation:

V. preference relations

(MCDM: social choice)

Aggregation for preference relations

- Social choice
  - studies voting rules, and how the preferences of a set of people can be aggregated to obtain the preference of the set.

Aggregation for preference relations

- Given preference relations, how aggregation is built?
  - Formalization of preferences with $>$ an $=$ (preference, indiference)
  - $F(R_1, R_2, \ldots, R_N)$ to denote aggregated preference

Aggregation for preference relations

- Given preference relations, how aggregation is built?
  - Formalization of preferences with $>$ an $=$ (preference, indiference)
  - $F(R_1, R_2, \ldots, R_N)$ to denote aggregated preference
  - Problems (I): consider
    - ⋆ $R^1 : x > y > z$
    - ⋆ $R^4 : y > z > x$
    - ⋆ $R^5 : z > x > y$

    $\rightarrow$ simple majority rule: $u > v$ if most prefer u to v

    - ⋆ $x > y, y > z, z > x$ (intransitive!!: $x > y, y > z$ but not $x > z$)
  - Problems (II):

    $\rightarrow$ Arrow impossibility theorem

Aggregation for preference relations

- Given preference relations, how aggregation is built?

- Axioms of Arrow impossibility theorem

  C0 Finite number of voters and more than one
  Number of alternatives more or equal to three

  C1 Universality: Voters can select any total preorder[1]

  C2 Transitivity: The result is a total preorder

  C3 Unanimity: If all agree on x better than y, then x better than y in the social preference

  C4 Independence of irrellevant alternatives: the social preference of x and y only depends on the preferences on x and y

  C5 No-dictatorship: No voter can be a dictatorship

- There is no function F that satisfies all C0-C5 axioms

[1] either $x \preceq y$ or $y \preceq x$

Aggregation for preference relations

- Given preference relations, how aggregation is built?
- Circumventing Arrow's theorem
  - Ignore the condition of universality
  - Ignore the condition of independence of irrelevant alternatives

Aggregation for preference relations

- Given preference relations, how aggregation is built?
  - Solutions failing the universality condition
    - ⋆ Simple peak, odd number of voters,
      Condorcet rule satisfies all other conditions

Aggregation for preference relations

- Given preference relations, how aggregation is built?
  - Solutions failing the condition of independence of irrelevant alternatives
    - ⋆ Condorcet rule with Copeland[2]:
    - ⋆ Borda count[3]

[2] Defined by Ramon Llull s. XIII
[3] Defined by Nicolas de Cusa s. XV.

Aggregation:

VI. Related topics

MCDM, utility functions, selection, Pareto

Aggregation

- Decision for utility functions
  Modelling, aggregation $= \mathbb{C}$, selection

	Seats	Security	Price	Comfort	trunk	$\mathbb{C} = AM$
Ford T	0	20	0	20	0	8
Seat 600	60	0	100	0	50	42
Simca 1000	100	30	100	50	70	70
VW	80	50	30	70	100	66
Citr. Acadiane	20	40	60	40	0	32

Aggregation

- MCDM: Aggregation to deal with contradictory criteria

Aggregation

- MCDM: Aggregation to deal with contradictory criteria
- But there are occasions in which ordering is clear

$$\text{when } a_i \leq b_i \text{ it is clear that } a \leq b$$

E.g.,

	Seats	Security	Price	Comfort	trunk	$\mathbb{C} = AM$
Seat 600	60	0	100	0	50	42
Simca 1000	100	30	100	50	70	70

Aggregation

- MCDM: Aggregation to deal with contradictory criteria
- But there are occasions in which ordering is clear

$$\text{when } a_i \leq b_i \text{ it is clear that } a \leq b$$

E.g.,

	Seats	Security	Price	Comfort	trunk	$\mathbb{C} = AM$
Seat 600	60	0	100	0	50	42
Simca 1000	100	30	100	50	70	70

- Pareto dominance: Given two vectors $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$, we say that b dominates a when $a_i \leq b_i$ for all i and there is at least one k such that $a_k < b_k$.

Aggregation for (numerical) utility functions

- Pareto set, Pareto frontier, or non dominance set:

	Seats	Security	Price	Comfort	trunk	$\mathbb{C} = AM$
Simca 1000	100	30	100	50	70	70
VW	80	50	30	70	100	66
Citr. Acadiane	20	40	60	40	0	32

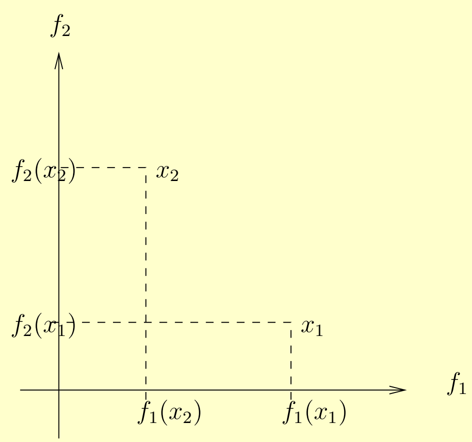
- Each one wins at least in one criteria to another one

Aggregation for (numerical) utility functions

- Pareto set, Pareto frontier, or non dominance set:
  Given a set of alternatives U represented by vectors $u = (u_1, \ldots, u_n)$, the Pareto frontier is the set $u \in U$ such that there is no other $v \in U$ such that v dominates u.

$$PF = \{u | \text{there is no } v \text{ s.t. } v \text{ dominates } u\}$$

- Pareto optimal: an element u of the Pareto set



Aggregation for (numerical) utility functions

- MCDM: we aggregate utility, and order according to utility
- Aggregation functions
  - Different aggregations lead to different orders
  - Aggregation establishes which points are *equivalent*
  - Different aggregations, establish different curves of points (level curves)

Criteria Satisfaction on:

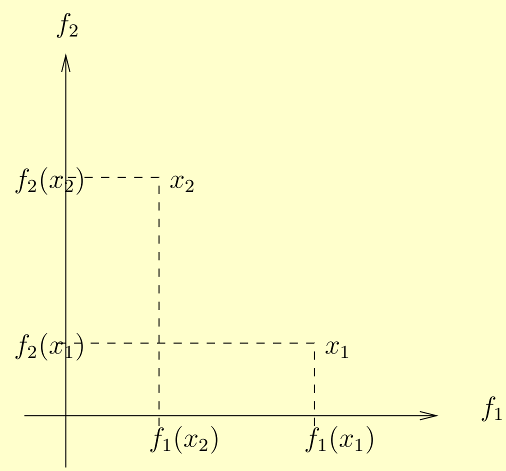
alt	Price	Quality	Comfort
FordT	0.2	0.8	0.3
206	0.7	0.7	0.8
...	...		

→

alt	Consensus
FordT	0.35
206	0.72
...	...

→

alt	Ranking
206	0.72
FordT	0.35
...	...



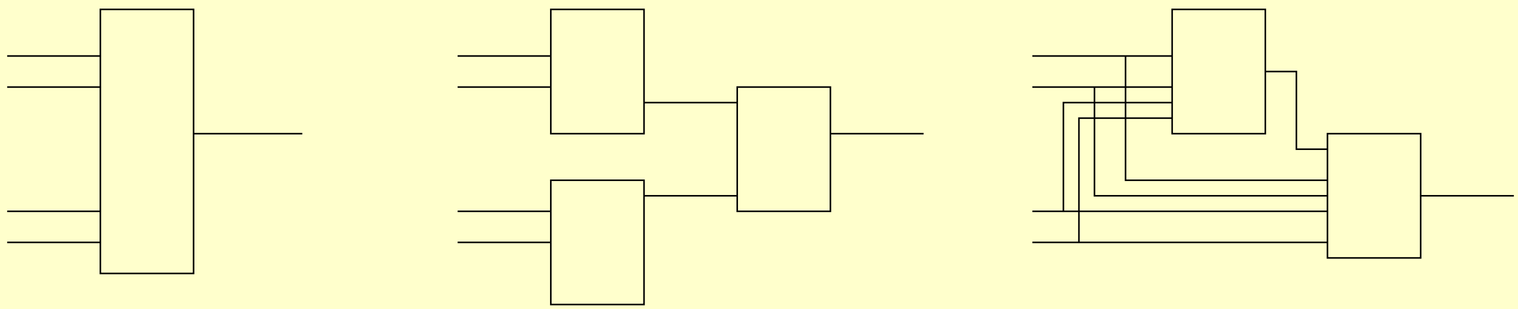
Aggregation:

VI. Related topics

Hierarchical models

Hierarchical Models for Aggregation

- Hierarchical model



- Properties. The following conditions hold

  (i) Every multistep Choquet integral is a monotone increasing, positively homogeneous, piecewise linear function.

  (ii) Every monotone increasing, positively homogeneous, piecewise linear function on a full-dimensional convex set in $\mathbb{R}^N$ is representable as a two-step Choquet integral such that the fuzzy measures of the first step are additive and the fuzzy measure of the second step is a 0-1 fuzzy measure.

Other related topics

- Aggregation functions
  - Functional equations (synthesis of judgements)
  - Fuzzy measures
  - Indices and evaluation methods
  - Model selection
- Decision making
  - Game theory (for decision making with adversary)
  - Decision under risk and uncertainty
  - Voting systems (social choice, aggregation of preferences)

Summary

Summary

- Aggregation functions
- There is life beyond the (weighted) mean
- Important concepts: the Pareto front (what is relevant to study)

Summary

- Aggregation functions
- There is life beyond the (weighted) mean
- Important concepts: the Pareto front (what is relevant to study)
- Fuzzy integrals to express non-independence
- Indices and methods to select functions and their parameters

Thank you

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