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Aggregation functions for social decision making

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Outline

Introduction

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Topic: Aggregation functions

- They are used in decision problems

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- They are used in decision problems
- To help establishing preferences for different pareto optimal situations

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MADM/MCDM: Select the best alternative from a set of alternatives

- Usually a finite set of alternatives (otherwise MODM)
- Each alternative evaluated in terms of a set of attributes (utilities)

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  Attributes (points of view/criteria) are in contradiction

Example: Decision making

- Criteria to order our car preferences: price, quality, and confort
  assign to each car $c_i \in Cars$ utility values $u_p(c_i), u_q(c_i), u_c(c_i)$
  assign importances to each criteria (or subset of criteria)
  (and combine values w.r.t. importances to find a global value (and order))
- Contradictory attributes: price vs. quality and confort

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Example: Ford T, Peugeot 308, Audi A4

	u_p	u_q	u_c	$\mathbb{C}$
Ford T	0.3	0.7	0.2	$\mathbb{C}(0.3, 0.7, 0.2)$
Peugeot 308	0.7	0.5	0.6	$\mathbb{C}(0.7, 0.5, 0.6)$
Audi A4	0.6	0.8	0.5	$\mathbb{C}(0.6, 0.8, 0.5)$

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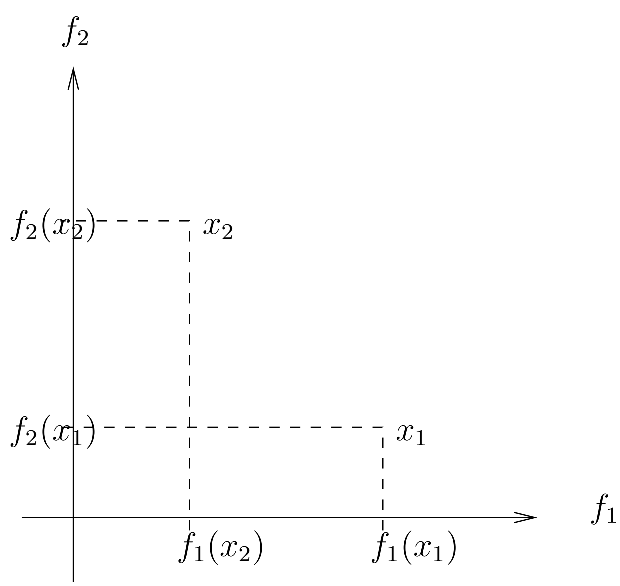
- Select alternatives with large utilities
  The larger the utility, the better
- However, in some cases, not possible to improve one criteria without worsening another
- Such solutions define the Pareto set



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Topic: Aggregation functions permit to order different pareto optimal solutions

- Different aggregation functions lead to different orderings
- Some functions

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- Different aggregation functions lead to different orderings
- Some functions
  - Arithmetic mean
  - Weighted mean
  - Ordered Weighting Averaging Operator
  - Choquet integral (integral for non-additive measures)
- Example: $\mathbb{C}$

	u_p	u_q	u_c	$\mathbb{C}$
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Aggregation Functions

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Aggregation Functions

Aggregation functions

- Weighting vector (dimension N): $v = (v_1 ... v_N)$ iff $v_i \in [0,1]$ and $\sum_i v_i = 1$
- Arithmetic mean (AM :$\mathbb{R}^N \to \mathbb{R}$): $AM(a_1, ..., a_N) = (1/N)\sum_{i=1}^{N} a_i$
- Weighted mean (WM: $\mathbb{R}^N \to \mathbb{R}$): $WM_{\mathbf{p}}(a_1, ..., a_N) = \sum_{i=1}^{N} p_i a_i$ ($\mathbf{p}$ a weighting vector of dimension N)
- Ordered Weighting Averaging operator (OWA: $\mathbb{R}^N \to \mathbb{R}$):

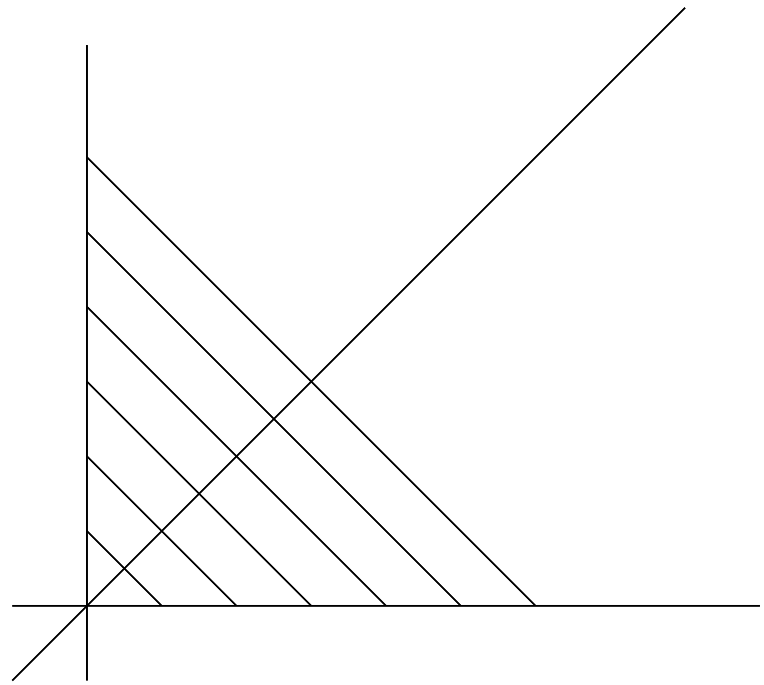
$$OWA_{\mathbf{w}}(a_1, ..., a_N) = \sum_{i=1}^{N} w_i a_{\sigma(i)},$$

where $\{\sigma(1), ..., \sigma(N)\}$ is a permutation of $\{1, ..., N\}$ s. t. $a_{\sigma(i-1)} \geq a_{\sigma(i)}$, and $\mathbf{w}$ a weighting vector.

Aggregation Functions

Aggregation functions: Arithmetic Mean (AM)

- Level curbes for Pareto Optimal solutions



Aggregation Functions

Aggregation functions: Weighted Mean (WM)

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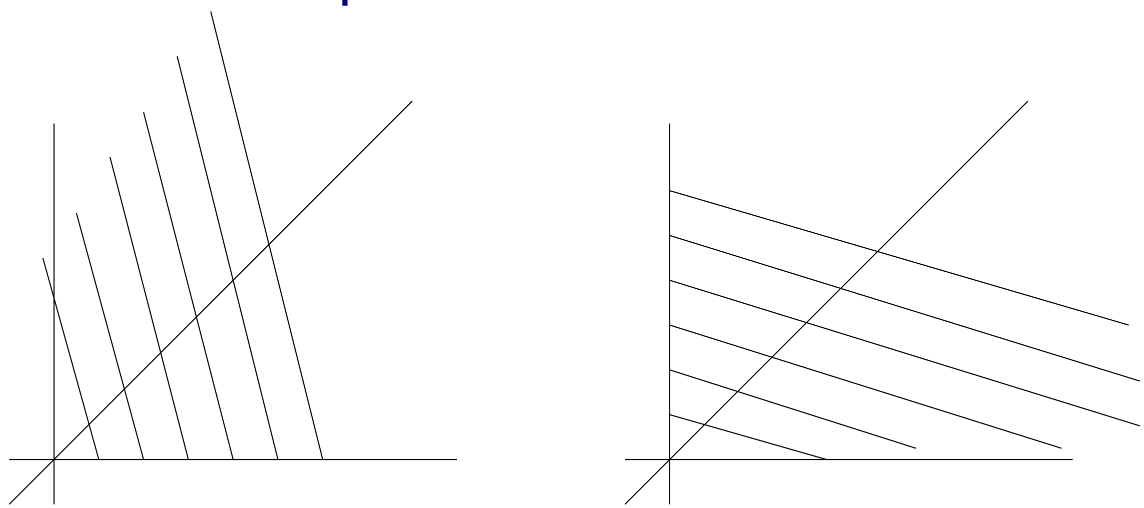


Aggregation Functions

Aggregation functions Interpretation of weights

Weights in WM and OWA: p and w

- Multicriteria Decision Making.
  p: importance of criteria,
  w: degree of compensation
- Fuzzy Constraint Satisfaction Problems.
  p: importance of the constraints,
  w: degree of compensation
- Robot Sensing (all data, same time instant).
  p: reliability of each sensor,
  w: importance of small values/outliers
- Robot Sensing (all data, different time instants).
  p: more importance to recent data than old one,
  w: importance of small values/outliers

Aggregation Functions

Weighted Ordered Weighted Averaging WOWA operator

(WOWA :$\mathbb{R}^N \rightarrow \mathbb{R}$):

$$WOWA_{\mathbf{p},\mathbf{w}}(a_1, ..., a_N) = \sum_{i=1}^N \omega_i a_{\sigma(i)}$$

where

$$\omega_i = w^*(\sum_{j \le i} p_{\sigma(j)}) - w^*(\sum_{j < i} p_{\sigma(j)}),$$

with σ a permutation of $\{1, ..., N\}$ s. t. $a_{\sigma(i-1)} \geq a_{\sigma(i)}$, and w^* a nondecreasing function that interpolates the points

$$\{(i/N, \sum_{j \le i} w_j)\}_{i=1,...,N} \cup \{(0,0)\}.$$

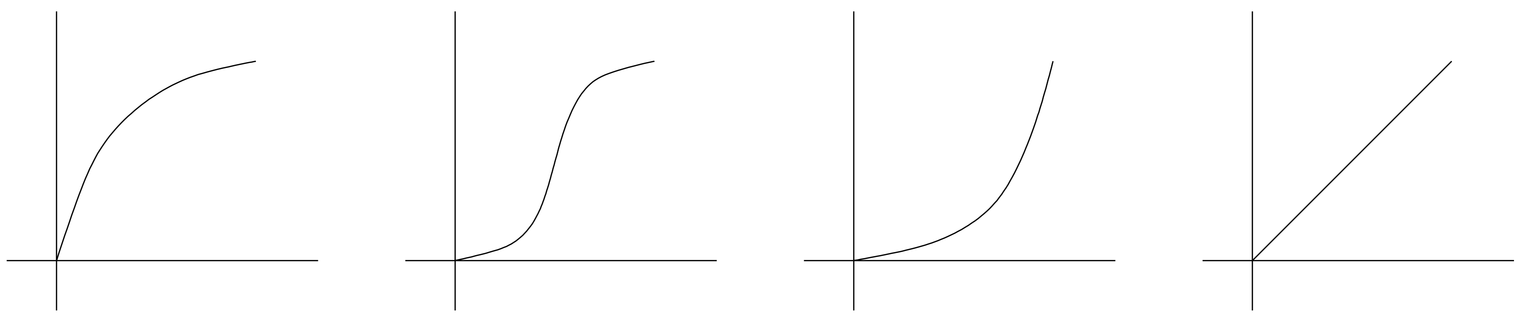
w^* is required to be a straight line when the points can be interpolated in this way.

Aggregation Functions

WOWA operator

The shape of the function w^* gives importance

- (a) to large values
- (b) to medium values
- (c) to small values
- (d) equal importance to all values



(a) (b) (c) (d)

Non-additive measures and integrals (Choquet integral)

Definitions

Additive measures: $(X, \mathcal{A})$ a measurable space; then, a set function μ is an additive measure if it satisfies

(i) $\mu(A) \geq 0$ for all $A \in \mathcal{A}$,

(ii) $\mu(X) \leq \infty$

(iii) $\mu(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$ for every countable sequence A_i $(i \geq 1)$ of $\mathcal{A}$ that is pairwise disjoint (i.e,. $A_i \cap A_j = \emptyset$ when $i \neq j$).

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Finite case: $\mu(A \cup B) = \mu(A) + \mu(B)$ for disjoint A, B

Definitions

Non-additive measures: $(X, \mathcal{A})$ a measurable space, a non-additive (fuzzy) measure μ on $(X, \mathcal{A})$ is a set function $\mu : \mathcal{A} \to [0, 1]$ satisfying the following axioms:

(i) $\mu(\emptyset) = 0$, $\mu(X) = 1$ (boundary conditions)
(ii) $A \subseteq B$ implies $\mu(A) \leq \mu(B)$ (monotonicity)

Differences

Non-additive measures vs. additive measures:

- In additive measures: $\mu(A) = \sum_{x \in A} p_x$

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- In additive measures: $\mu(A) = \sum_{x \in A} p_x$
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  $\rightarrow$ three cases possible
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  - $\mu(A) = \sum_{x \in A} p_x$ (no interaction)

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  - $\mu(A) = \sum_{x \in A} p_x$ (no interaction)
  - $\mu(A) < \sum_{x \in A} p_x$ (negative interaction)
  - $\mu(A) > \sum_{x \in A} p_x$ (positive interaction)

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- In our example:

Differences

Non-additive measures vs. additive measures:

- Is non-additivity useful ?
  Yes
- In our example:
  - $\mu(\{price\})$, $\mu(\{quality\})$, $\mu(\{confort\})$
  - $\mu(\{price, quality\})$, $\mu(\{price, confort\})$, $\mu(\{quality, confort\})$
  - $\mu(\{price, quality, confort\})$

Number of parameters

Non-additive measures vs. additive measures:

- How to define an additive measure on X?

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Non-additive measures vs. additive measures:

- How to define an additive measure on X?
  One probability value for each element
  $\rightarrow |X|$ values
- How to define a non-additive measure?
  One value for each set
  $\rightarrow 2^{|X|}$ values

What can we do with a measure?

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    $\rightarrow$ expectation: $\sum p_x f(x)$
    $\longrightarrow$ Lebesgue integral (continuous case: $\int f dp$)
  - Integral w.r.t. non-additive measure μ
    $\rightarrow$ expectation like

$$\sum_{i=1}^{N} f(x_{\sigma(i)})[\mu(A_{\sigma(i)}) - \mu(A_{\sigma(i-1)})]$$

$\longrightarrow$ Choquet integral (continuous case: $(C) \int f d\mu$)

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The Choquet integral is a Lebesgue integral when the measure is additive

Choquet integrals

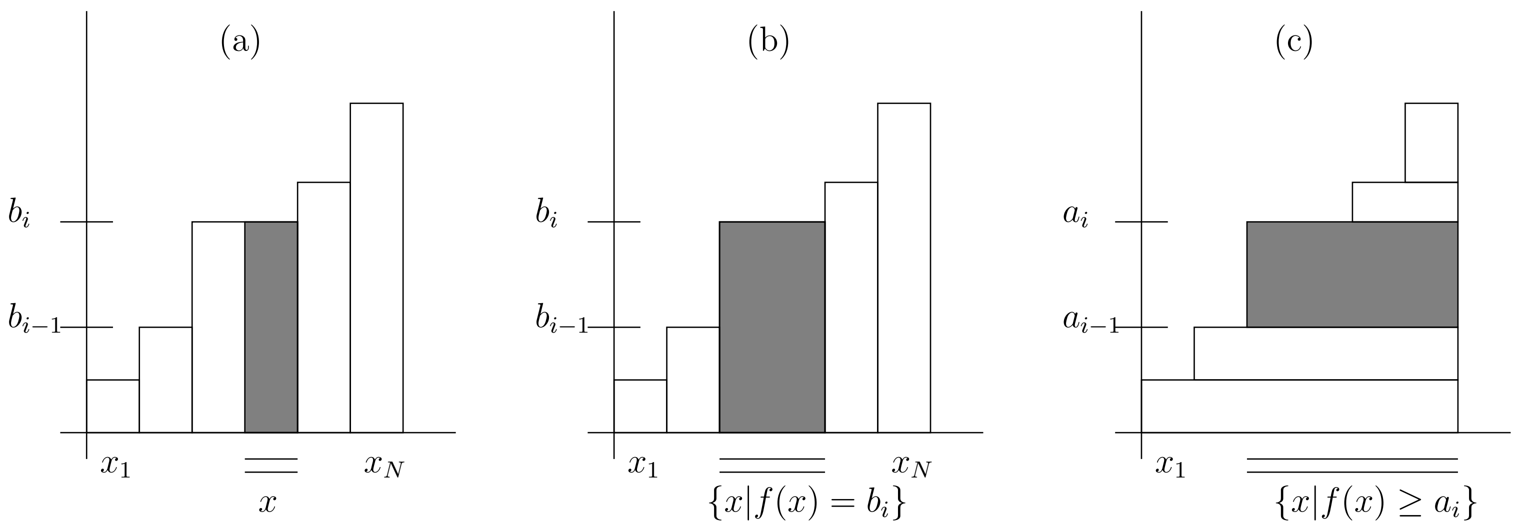
Integral: $\int f d\mu =$ (for additive measures)

(6.5) $\sum_{x \in X} f(x)\mu(\{x\})$

(6.6) $\sum_{i=1}^{R} b_i \mu(\{x | f(x) = b_i\})$

(6.7) $\sum_{i=1}^{N} (a_i - a_{i-1})\mu(\{x | f(x) \geq a_i\})$

(6.8) $\sum_{i=1}^{N} (a_i - a_{i-1})\big(1 - \mu(\{x | f(x) \leq a_{i-1}\})\big)$

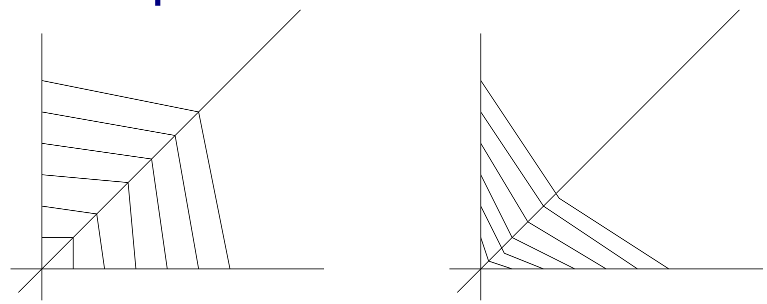


Among (6.5), (6.6) and (6.7), only (6.7) satisfies internality.

Choquet integrals

Aggregation functions: Choquet integral (CI)

- Level curbes for Pareto Optimal solutions



Application (a paradox)

Example

Decision making:

Ellsberg paradox (Ellsberg, 1961), an urn, 90 balls ...

Color of balls	Red	Black	Yellow
Number of balls	30	60	
f_R	\$ 100	0	0
f_B	\$ 0	\$ 100	0
f_{RY}	\$ 100	0	\$ 100
f_{BY}	\$ 0	\$ 100	\$ 100

- Usual (most people's) preferences
  - $f_B \prec f_R$

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- Usual (most people's) preferences
  - $f_B \prec f_R$
  - $f_{RY} \prec f_{BY}$
- No solution exist with additive measures, but can be solved with non-additive ones

Distorted Probabilities

Distorted Probabilities: introduction

An open question:

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Non-additive measures vs. additive measures:

How to define a non-additive measure?

One value for each set

$\rightarrow 2^{|X|}$ values

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- Compact representation of non-additive measures:

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A possible solution:

Distorted Probabilities.

- Compact representation of non-additive measures:
  - Only $|X|$ values (a probability) and a function (distorting function)

Distorted Probabilities: Definition

- Representation of a fuzzy measure:
  - f and P represent a fuzzy measure μ, iff

$$\mu(A) = f(P(A)) \text{ for all } A \in 2^X$$

    f a real-valued function, P a probability measure on $(X, 2^X)$
  - f is strictly increasing w.r.t. a probability measure P iff $P(A) < P(B)$ implies $f(P(A)) < f(P(B))$
  - f is nondecreasing w.r.t. a probability measure P iff $P(A) < P(B)$ implies $f(P(A)) \leq f(P(B))$

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  - μ is a distorted probability if μ is represented by a probability distribution P and a function f nondecreasing w.r.t. a probability P.

Distorted Probabilities: Definition

- Representation of a fuzzy measure: distorted probability
  - μ is a distorted probability if μ is represented by a probability distribution P and a function f nondecreasing w.r.t. a probability P.
- So, for a given reference set X we need:
  - Probability distribution on X: $p(x)$ for all $x \in X$
  - Distortion function f on the probability measure: $f(P(A))$

The End

m-dimensional Distorted Probabilities

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Justification: Why any extension of distorted probabilities?

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- The number of distorted probabilities.

m-Dimensional Distorted Probabilities

Justification: Why any extension of distorted probabilities?

- The number of distorted probabilities.

  Observe the following
  - For $X = \{1, 2, 3\}$, 2/8 of distorted probabilities.
  - For larger sets X ...

    ... the proportion of distorted probabilities decreases rapidly
  - For $\mu(\{1\}) \leq \mu(\{2\}) \leq \ldots$

$\lvert X \rvert$	Number of possible orderings for Distorted Probabilities	Number of possible orderings for Fuzzy Measures
1	1	1
2	1	1
3	2	8
4	14	70016
5	546	$\mathcal{O}(10^{12})$
6	215470	–

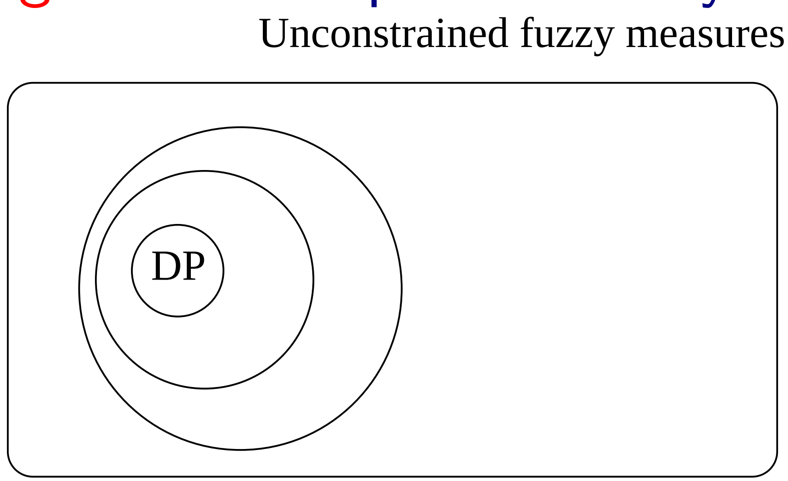
m-Dimensional Distorted Probabilities

Justification: Why any extension of distorted probabilities?

The number of distorted probabilities.

Goal:

- To cover a larger region of the space of fuzzy measures



→ (similar to the property of k-additive fuzzy measures)

$$DP_{1,X} \subset DP_{2,X} \subset DP_{3,X} \cdots \subset DP_{|X|,X}$$

m-Dimensional Distorted Probabilities

- In distorted probabilities:
  - One probability distribution
  - One function f to distort the probabilities
- Extension to:
  - m probability distributions
  - One function f to distort/combine the probabilities

m-Dimensional Distorted Probabilities

- In distorted probabilities:
  - One probability distribution
  - One function f to distort the probabilities
- Extension to:
  - m probability distributions P_i
    - ⋆ Each P_i defined on X_i
    - ⋆ Each X_i is a partition element of X (a dimension)
  - One function f to distort/combine the probabilities

m-Dimensional Distorted Probabilities: Example

- Running example:
  - A fuzzy measure that is not a distorted probability:

    $\mu(\emptyset) = 0$ $\quad$ $\mu(\{M, L\}) = 0.9$

    $\mu(\{M\}) = 0.45$ $\quad$ $\mu(\{P, L\}) = 0.9$

    $\mu(\{P\}) = 0.45$ $\quad$ $\mu(\{M, P\}) = 0.5$

    $\mu(\{L\}) = 0.3$ $\quad$ $\mu(\{M, P, L\}) = 1$
  - Partition on X:
    - $\star$ $X_1 = \{L\}$ (Literary subjects)
    - $\star$ $X_2 = \{M, P\}$ (Scientific Subjects)

m-Dimensional Distorted Probabilities: Definition

- m-dimensional distorted probabilities.
  - μ is an at most m dimensional distorted probability if

$$\mu(A) = f(P_1(A \cap X_1), P_2(A \cap X_2), \cdots, P_m(A \cap X_m))$$

where,
$\{X_1, X_2, \cdots, X_m\}$ is a partition of X,
P_i are probabilities on $(X_i, 2^{X_i})$,
f is a function on $\mathbb{R}^m$ strictly increasing with respect to the i-th axis for all $i = 1, 2, \ldots, m$.

- μ is an m-dimensional distorted probability if it is an at most m dimensional distorted probability but it is not an at most $m-1$ dimensional.

m-Dimensional Distorted Probabilities: Example

- Running example: a two dimensional distorted probability

$$\mu(A) = f(P_1(A \cap \{L\}), P_2(A \cap \{M, P\}))$$

  - with partition on $X = \{M, L, P\}$
    1. Literary subject $\{L\}$
    2. Science subjects $\{M, P\}$,
  - probabilities
    1. $P_1(\{L\}) = 1$
    2. $P_2(\{M\}) = P_2(\{P\}) = 0.5$,
  - and distortion function f defined by

1	{L}	0.3	0.9	1.0
0	∅	0	0.45	0.5
	sets	∅	{M}, {P}	{M,P}
f		∅	0.5	1

Distorted Probabilities and Multisets

an approach to define (simple) fuzzy measures on multisets

Distorted Probabilities and Multisets

Multisets: elements can appear more than once

- Defined in terms of $count_M : X \to \{0\} \cup \mathbb{N}$
  e.g. when $X = \{a, b, c, d\}$ and $M = \{a, a, b, b, c, c, c\}$,
  $count_M(a) = 2$, $count_M(b) = 3$, $count_M(c) = 3$, $count_M(d) = 0$.
- A and B multisets on X, then
  - $A \subseteq B$ if and only if $count_A(x) \leq count_B(x)$ for all x in X (used to define submultiset).
  - $A \cup B$:
    $count_{A \cup B}(x) = \max(count_A(x), count_B(x))$ for all x in X.
  - $A \cap B$:
    $count_{A \cap B}(x) = \min(count_A(x), count_B(x))$ for all x in X.

Distorted Probabilities and Multisets

Fuzzy measure on multiset: X a reference set, M a multiset on X s.t. $M \neq \emptyset$; then, the function μ from $(M, \mathcal{P}(M))$ to $[0, 1]$ is a fuzzy measure if the following holds:

- $\mu(\emptyset) = 0$ and $\mu(M) = 1$
- $\mu(A) \leq \mu(B)$ when $A \subseteq B$ and $B \subseteq M$.

Distorted Probabilities and Multisets

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How to define fuzzy measures?:

Distorted Probabilities and Multisets

Fuzzy measure on multiset: X a reference set, M a multiset on X s.t. $M \neq \emptyset$; then, the function μ from $(M, \mathcal{P}(M))$ to $[0, 1]$ is a fuzzy measure if the following holds:

- $\mu(\emptyset) = 0$ and $\mu(M) = 1$
- $\mu(A) \leq \mu(B)$ when $A \subseteq B$ and $B \subseteq M$.

How to define fuzzy measures?:

- Even more parameters $\prod_{x \in X} count_M(x)$!!

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We present two alternative (but related) approaches

Distorted Probabilities and Multisets

1st approach: Definition based on a pseudoadditive integral:
Nondecreasing function-based fuzzy measures

- X a reference set, M a multiset on X and μ a $\oplus$-decomposable fuzzy measure on X. Let $f : [0, \infty) \to [0, \infty)$ be a non-decreasing function with $f(0) = 0$ and $f(m(M)) = 1$. Then, we define a fuzzy measure ν on $\mathcal{P}(M)$ by

$$\nu_f(A) = f(m(A))$$

where m is the multiset function $m : \mathcal{P}(M) \to [0, \infty)$ defined by

$$m(A) = (D) \int count_A d\mu.$$

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- Rationale of the definition: $(C) \int \chi_A d\mu = \mu(A)$
- Properties:

if $A \subseteq B$ by the monotonicity of the integral $m(A) \leq m(B)$
$\rightarrow$ monotonicity condition of the fuzzy measure fulfilled

Distorted Probabilities and Multisets

2nd approach: Definition based on prime numbers[1]:

- Define
$$n(A) := \prod_{x \in X} \phi(x)^{count_A(x)},$$
where ϕ is an injective function from X to the prime numbers, and let h be a non-decreasing function from $\mathbb{N}$ to $[0, 1]$ satisfying $h(1) = 0$ and $h(n(M)) = 1$. We define the prime number-based fuzzy measure
$$\nu_{\phi,h}(A) = h(n(A)).$$

[1] and using the unique factorization of integers into prime numbers

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$$\nu_{\phi,h}(A) = h(n(A)).$$

Properties:

if $A \neq B$ by the unique factorization $n(A) \neq n(B)$

if $A \subseteq B$ by the factorization $n(A) < n(B)$

$\rightarrow$ monotonicity condition of the fuzzy measure fulfilled

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Distorted Probabilities and Multisets

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  - f and the distortion
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- Fuzzy measures based on prime-number are a particular case of the 1st approach
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- It seems that there is some parallelism between prime-number based fuzzy measures and distorted probabilities
  - f and the distortion
  - ϕ and the probability distribution
- Can we establish a relationship??

Results

Properties:

- ν a fuzzy measure according to Approach 1 on a proper finite set $(M, \mathcal{P}(M)) = (X, 2^X)$. Then ν is a distorted probability on $(X, 2^X)$.

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- Same for Approach 2 (primer-number definition)
- This is easy to prove (consists on defining the probability distribution)
- So, Approach 1 and Approach 2 equal to or more general than distorted probabilities

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Properties:

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  $\rightarrow$ probabilities and prime numbers play the same role

  $\rightarrow \sum_{x \in A} p_x$ and $\prod_{x \in A} \phi(x)$ play the same role

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  Surprising corollary: Approach 1 and approach 2 are equivalent.

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Proof based on some results on number theory about the existence of k prime numbers in certain intervals (Bertrand's postulate).

Results

An example to satisfy curiosity:

- μ distorted probability $p = (0.05, 0.1, 0.2, 0.3, 0.35)$, $g(x) = x^2$.

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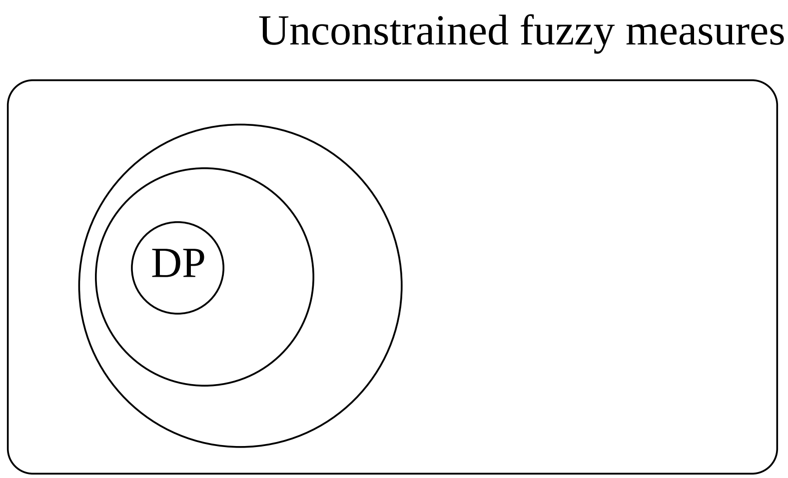
- μ distorted probability $p = (0.05, 0.1, 0.2, 0.3, 0.35)$, $g(x) = x^2$.
- Representation with prime numbers and appropriate function

$$
\begin{aligned}
\phi(x_1) &= 17 \in [16.0, 32.0001] \\
\phi(x_2) &= 367 \in [362.041, 724.081] \\
\phi(x_3) &= 185369 \in [185366.0, 370732.0] \\
\phi(x_4) &= 94907801 \in [9.49078 \times 10^7, 1.89816 \times 10^8] \\
\phi(x_5) &= 2147524151 \in [2.14752 \times 10^9, 4.29505 \times 10^9]
\end{aligned}
$$

m-Dimensional DP for multisets

How to solve the *problem* that not all fuzzy measures for multisets are distorted probabilities ?

- Same approach as before: m-dimensional prime number-based fuzzy measure



m-Dimensional DP for multisets

m-dimensional prime number-based fuzzy measure

- μ is an at most m-dimensional prime number-based fuzzy measure if

$$\mu(A) = f(n_1(A \cap X_1), \ldots, n_m(A \cap X_m))$$

where,
$\{X_1, X_2, \cdots, X_m\}$ is a partition of X,
$n_i(A) = \prod_{x \in X_i} \phi(x)^{count_A(x)}$ with ϕ_i injective functions from X_i to the prime numbers
f is a strictly increasing function with respect to the i-th axis for all $i = 1, 2, \ldots, m$.

μ is an m-dimensional prime number-based fuzzy measure if it is an at most m dimensional distorted probability but it is not an at most $m-1$ dimensional.

m-Dimensional DP for multisets

Properties:

- All fuzzy measures are at most $|X|$-dimensional prime number-based fuzzy measures.

Integral

Integral

Definition

- Boundary measures:
  - $\mu^+(A) = A \cdot M$ for all $A \subseteq X$
  - $\mu_-(A) = A \cap M$ for all $A \subseteq X$
- They satisfy:

$$\mu_-(A) \leq \mu^+(A)$$

  and, therefore,

$$(C)\int f d\mu_- < (C)\int f d\mu^+$$

Summary

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- Brief justification of the use of non-additive (fuzzy) measures
- Introduction to distorted probabilities
- Extensions
  - m-dimensional distorted probabilities
  - Fuzzy measures for multisets