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On some extensions and applications of non-additive measures

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A (short) motivation

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Topic: Non-additive (fuzzy) measures

- A generalization of additive measures (probabilities)

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- A generalization of additive measures (probabilities)
- Equivalent terms: non-additive measures, fuzzy measures, capacities

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Why these measures are studied?

- Mathematical interest
- Applications

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    - ⋆ Measures and distances (e.g. entropy / Hellinger distance)

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  - Constructions
    - ⋆ Integrals with respect to these measures (e.g. Choquet integrals)

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    - ⋆ Subjective evaluation
    - ⋆ Data fusion
    - ⋆ Computer vision
    - ⋆ Distances

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    - ⋆ Computer vision
    - ⋆ Distances

      → a common theme:
      to take into account interactions

      → a common advantage:
      more expressive power than with additive models

A short motivation

Why these measures are studied?

- Decision making
  - Criteria to order our car preferences: price, quality, and confort
    assign to each car $c_i \in Cars$ utility values $u_p(c_i), u_q(c_i), u_c(c_i)$
    assign importances to each criteria (or subset of criteria)
    and combine values w.r.t. importances to find a global value (and order)
- Data fusion
  - Sensors give distances to the nearest object: s_1, s_2, s_3
    assign importances to sensors (or subsets of sensors)
    and combine values w.r.t. importances to find a reliable value

A short motivation

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$\rightarrow$ distorted probabilities as a compact representation of (some) non-additive measure
$\longrightarrow$ useful for applications

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  - $\perp$-decomposable fuzzy measures ($\mu(A \cup B) = \mu(A) \oplus \mu(B)$)
  - Sugeno λ-measures ($\mu(A \cup B) = \mu(A) + \mu(B) + \lambda\mu(A)\mu(B)$)
  - k-additive fuzzy measures (in terms of the Möbius transform)
  - Hierarchically decomposable fuzzy measures ($\oplus_i$ + hierarchy of X)

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Distorted probability: simple

Outline

1. Short Motivation
2. Introduction (definitions)
3. Applications (short)
4. Distorted Probabilities
5. m-dimensional Distorted Probabilities
6. Distorted Probabilities and Multisets
7. Summary

Introduction

Definitions

Additive measures: $(X, \mathcal{A})$ a measurable space; then, a set function μ is an additive measure if it satisfies

(i) $\mu(A) \geq 0$ for all $A \in \mathcal{A}$,
(ii) $\mu(X) \leq \infty$
(iii) $\mu(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$ for every countable sequence A_i $(i \geq 1)$ of $\mathcal{A}$ that is pairwise disjoint (i.e,. $A_i \cap A_j = \emptyset$ when $i \neq j$).

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Finite case: $\mu(A \cup B) = \mu(A) + \mu(B)$ for disjoint A, B

Definitions

Non-additive measures: $(X, \mathcal{A})$ a measurable space, a non-additive (fuzzy) measure μ on $(X, \mathcal{A})$ is a set function $\mu : \mathcal{A} \rightarrow [0, 1]$ satisfying the following axioms:

(i) $\mu(\emptyset) = 0$, $\mu(X) = 1$ (boundary conditions)
(ii) $A \subseteq B$ implies $\mu(A) \leq \mu(B)$ (monotonicity)

Differences

Non-additive measures vs. additive measures:

- In additive measures: $\mu(A) = \sum_{x \in A} p_x$

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  - $\mu(A) < \sum_{x \in A} p_x$ (negative interaction)
  - $\mu(A) > \sum_{x \in A} p_x$ (positive interaction)

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- How to define an additive measure on X?
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- How to define a non-additive measure?
  One value for each set
  $\rightarrow 2^{|X|}$ values

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    $\rightarrow$ expectation
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    $\longrightarrow$ Lebesgue integral (continuous case: $\int f dp$)
  - Integral w.r.t. non-additive measure μ
    $\rightarrow$ expectation like
    $$\sum_{i=1}^{N} f(x_{\sigma(i)})[\mu(A_{\sigma(i)}) - \mu(A_{\sigma(i-1)})]$$
    $\longrightarrow$ Choquet integral (continuous case: $(C) \int f d\mu$)

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The Choquet integral is a Lebesgue integral when the measure is additive

Applications

Example

Decision making:

Ellsberg paradox (Ellsberg, 1961), an urn, 90 balls ...

Color of balls	Red	Black	Yellow
Number of balls	30	60	
f_R	\$ 100	0	0
f_B	\$ 0	\$ 100	0
f_{RY}	\$ 100	0	\$ 100
f_{BY}	\$ 0	\$ 100	\$ 100

- Usual (most people's) preferences
  - $f_B \prec f_R$

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- Usual (most people's) preferences
  - $f_B \prec f_R$
  - $f_{RY} \prec f_{BY}$
- No solution exist with additive measures, but can be solved with non-additive ones

Applications

Subjective Evaluation:

Subjective evaluation and application field for non-additive (fuzzy) measures from the beginning.
(Sugeno, 1974, p.2):

> "The purposes of this dissertation are to propose the concept of fuzzy measures and integrals [11,12] as a way for expressing human subjectivity and to discuss their applications."

Applications

Definition of Subjective Evaluation: (Dubois and Prade, 1997)

> "Formally speaking, the subjective evaluation problem can be viewed as the synthesis, the identification of a function which maps the attribute values describing the situation to evaluate into a discrete domain (classification), or a continuous one (absolute evaluation). More generally, we may look for the degree of membership of the situation to a category, or have a function yielding a fuzzy evaluation. This function is in general not available as such, but is implicitly, and partially, described in terms of criteria, or by means of expert rules, or through some fuzzy algorithm. It may also happen that the function is only partially known by exemplification through prototypical examples of situations for which the evaluation is available."

Example

Example: (Grabish, 1995) Evaluation of students

- students (A, B, C) on three subjects (M, P, L)
  Ada, Byron, Countess; maths, physics, literature
- Marks:

Student		M	P	L
Ada	f_A	18	16	10
Byron	f_B	10	12	18
Countess	f_C	14	15	15

- Preferences:
  - Assign the same weight to mathematics and physics, and more weight to this subjects than to literature.
  - Represent the following preference on the students:

$$B \prec A \prec C.$$

Example

Example: (Grabish, 1995)

- No solution with additive measures
  We can use non-additive measures (with the Choquet integral)

Distorted Probabilities

Distorted Probabilities: introduction

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Non-additive measures vs. additive measures:

How to define a non-additive measure?

One value for each set

$\rightarrow 2^{|X|}$ values

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Distorted Probabilities.

- Compact representation of non-additive measures:

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A possible solution:

Distorted Probabilities.

- Compact representation of non-additive measures:
  - Only $|X|$ values (a probability) and a function (distorting function)

Distorted Probabilities: Definition

- Representation of a fuzzy measure:
  - f and P represent a fuzzy measure μ, iff

$$\mu(A) = f(P(A)) \text{ for all } A \in 2^X$$

  f a real-valued function, P a probability measure on $(X, 2^X)$
  - f is strictly increasing w.r.t. a probability measure P iff $P(A) < P(B)$ implies $f(P(A)) < f(P(B))$
  - f is nondecreasing w.r.t. a probability measure P iff $P(A) < P(B)$ implies $f(P(A)) \leq f(P(B))$

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  - μ is a distorted probability if μ is represented by a probability distribution P and a function f nondecreasing w.r.t. a probability P.

Distorted Probabilities: Definition

- Representation of a fuzzy measure: distorted probability
  - μ is a distorted probability if μ is represented by a probability distribution P and a function f nondecreasing w.r.t. a probability P.
- So, for a given reference set X we need:
  - Probability distribution on X: $p(x)$ for all $x \in X$
  - Distortion function f on the probability measure: $f(P(A))$

Distorted Probabilities: Application

- Given a distorted probability ...
  ... we can apply any fuzzy integral

- E.g.
  - the Choquet integral
  - the Sugeno integral

Distorted Probabilities: Application

- Distorted probability and Choquet integral:
  - The WOWA operator can be represented as a Choquet integral with a distorted probability.
    - ⋆ WOWA generalizes both the WM and the OWA, using both WM weights and OWA weights.
  - From the distorted probability perspective, in WOWA:
    - ⋆ the WM weights correspond to the probability distribution
    - ⋆ the OWA weights are used to the construct the distortion function

Distorted Probabilities: Properties

- Some distorted probabilities are not decomposable fuzzy measures.
- Some distorted probabilities cannot be represented easily with other families of fuzzy measures $\rightarrow$ they really belong to another family.

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- 1st. example (I):
  - μ on $X = \{a, b, c\}$ with $p(a) = 0.2$, $p(b) = 0.35$, $p(c) = 0.45$, and

$$f(x) = \begin{cases} 0 & \text{if } x < 0.5 \\ 0.2 & \text{if } 0.5 \le x < 0.6 \\ 0.4 & \text{if } 0.6 \le x < 0.85 \\ 1.0 & \text{if } 0.85 \le x \le 1.0 \end{cases}$$

Distorted Probabilities: Properties

- Some distorted probabilities are not decomposable fuzzy measures.
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- 1st. example (II):
  - $\mu(\emptyset) = 0$, $\mu(\{a\}) = 0$, $\mu(\{b\}) = 0$, $\mu(\{c\}) = 0$,
    $\mu(\{a, b\}) = 0.2$, $\mu(\{a, c\}) = 0.4, \mu(\{b, c\}) = 0.4, \mu(\{a, b, c\}) = 1$
  - μ is a DP but not a $\perp$-decomposable fuzzy measure
    because, there is no t-conorm s.t. $\perp(0, 0) \neq 0$
    $\rightarrow$ as $\mu(\{a, b\}) = 0.2$ when $\mu(\{a\}) = 0$ and $\mu(\{b\}) = 0$,
    we would require $0.2 = \mu(\{a, b\}) = \perp(\mu(\{a\}), \mu(\{b\})) = \perp(0, 0)$.

Distorted Probabilities: Properties

- Some distorted probabilities are not decomposable fuzzy measures.
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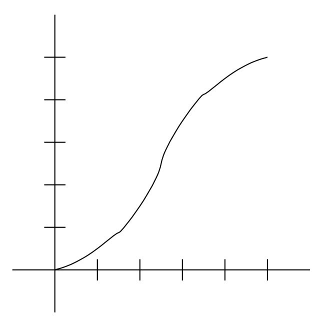
- 2nd. example (I):
  - $\mu_{\mathbf{p},\mathbf{w}}$ over $X = \{x_1, x_2, x_3, x_4, x_5\}$ from (probability distribution) $\mathbf{p} = (0.2, 0.3, 0.1, 0.2, 0.1)$, and function (from $\mathbf{w} = (0.1, 0.2, 0.4, 0.2, 0.1)$):



Distorted Probabilities: Properties

- Some distorted probabilities are not decomposable fuzzy measures.
- Some distorted probabilities cannot be represented easily with other families of fuzzy measures.
- 2nd. example (II):
  - $\mu_{\mathbf{p},\mathbf{w}}$ is a 5-additive fuzzy measure because $m(A) \neq 0$ for all A.
  - E.g.,
    $m(\{x_1, x_2, x_3, x_4, x_5\}) = 0.50746528$,
    $m(\{x_1, x_2, x_3, x_4\}) = -0.2537326$.
    There is no k-additive fuzzy measure equivalent to $\mu_{\mathbf{p},\mathbf{w}}$ for $k < 5$.

m-dimensional Distorted Probabilities

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Justification: Why any extension of distorted probabilities?

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The number of distorted probabilities.

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The number of distorted probabilities.

Observe the following

- For $X = \{1, 2, 3\}$, 2/8 of distorted probabilities.
- For larger sets X ...
  ... the proportion of distorted probabilities decreases rapidly
- For $\mu(\{1\}) \leq \mu(\{2\}) \leq \ldots$

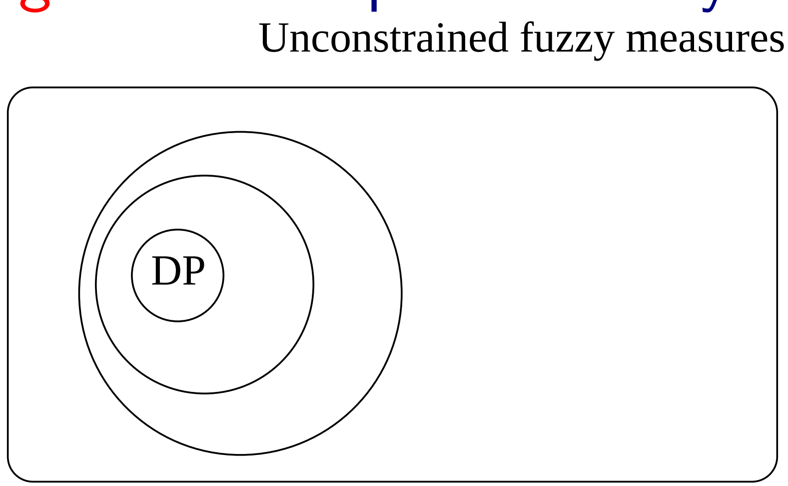
$\lvert X \rvert$	Number of possible orderings for Distorted Probabilities	Number of possible orderings for Fuzzy Measures
1	1	1
2	1	1
3	2	8
4	14	70016
5	546	$\mathcal{O}(10^{12})$
6	215470	–

m-Dimensional Distorted Probabilities

Justification: Why any extension of distorted probabilities?
The number of distorted probabilities.

Goal:

- To cover a larger region of the space of fuzzy measures



→ (similar to the property of k-additive fuzzy measures)

$$DP_{1,X} \subset DP_{2,X} \subset DP_{3,X} \cdots \subset DP_{|X|,X}$$

m-Dimensional Distorted Probabilities

- In distorted probabilities:
  - One probability distribution
  - One function f to distort the probabilities
- Extension to:
  - m probability distributions
  - One function f to distort/combine the probabilities

m-Dimensional Distorted Probabilities

- In distorted probabilities:
  - One probability distribution
  - One function f to distort the probabilities
- Extension to:
  - m probability distributions P_i
    - ⋆ Each P_i defined on X_i
    - ⋆ Each X_i is a partition element of X (a dimension)
  - One function f to distort/combine the probabilities

m-Dimensional Distorted Probabilities: Example

- Running example:
  - A fuzzy measure that is not a distorted probability:

$$\begin{array}{ll} \mu(\emptyset) = 0 & \mu(\{M, L\}) = 0.9 \\ \mu(\{M\}) = 0.45 & \mu(\{P, L\}) = 0.9 \\ \mu(\{P\}) = 0.45 & \mu(\{M, P\}) = 0.5 \\ \mu(\{L\}) = 0.3 & \mu(\{M, P, L\}) = 1 \end{array}$$

  - Partition on X:
    - $\star$ $X_1 = \{L\}$ (Literary subjects)
    - $\star$ $X_2 = \{M, P\}$ (Scientific Subjects)

m-Dimensional Distorted Probabilities: Definition

- m-dimensional distorted probabilities.
  - μ is an at most m dimensional distorted probability if

$$\mu(A) = f(P_1(A \cap X_1), P_2(A \cap X_2), \cdots, P_m(A \cap X_m))$$

where,
$\{X_1, X_2, \cdots, X_m\}$ is a partition of X,
P_i are probabilities on $(X_i, 2^{X_i})$,
f is a function on $\mathbb{R}^m$ strictly increasing with respect to the i-th axis for all $i = 1, 2, \ldots, m$.

- μ is an m-dimensional distorted probability if it is an at most m dimensional distorted probability but it is not an at most $m - 1$ dimensional.

m-Dimensional Distorted Probabilities: Example

- Running example: a two dimensional distorted probability

$$\mu(A) = f(P_1(A \cap \{L\}), P_2(A \cap \{M, P\}))$$

  - with partition on $X = \{M, L, P\}$
    1. Literary subject $\{L\}$
    2. Science subjects $\{M, P\}$,
  - probabilities
    1. $P_1(\{L\}) = 1$
    2. $P_2(\{M\}) = P_2(\{P\}) = 0.5$,
  - and distortion function f defined by

1	{L}	0.3	0.9	1.0
0	∅	0	0.45	0.5
	sets	∅	{M}, {P}	{M,P}
f		∅	0.5	1

Distorted Probabilities and Multisets

an approach to define (simple) fuzzy measures on multisets

Distorted Probabilities and Multisets

Multisets: elements can appear more than once

- Defined in terms of $count_M : X \to \{0\} \cup \mathbb{N}$
  e.g. when $X = \{a, b, c, d\}$ and $M = \{a, a, b, b, c, c, c\}$,
  $count_M(a) = 2$, $count_M(b) = 3$, $count_M(c) = 3$, $count_M(d) = 0$.
- A and B multisets on X, then
  - $A \subseteq B$ if and only if $count_A(x) \leq count_B(x)$ for all x in X (used to define submultiset).
  - $A \cup B$:
    $count_{A \cup B}(x) = \max(count_A(x), count_B(x))$ for all x in X.
  - $A \cap B$:
    $count_{A \cap B}(x) = \min(count_A(x), count_B(x))$ for all x in X.

Distorted Probabilities and Multisets

Fuzzy measure on multiset: X a reference set, M a multiset on X s.t. $M \neq \emptyset$; then, the function μ from $(M, \mathcal{P}(M))$ to $[0,1]$ is a fuzzy measure if the following holds:

- $\mu(\emptyset) = 0$ and $\mu(M) = 1$
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We present two alternative (but related) approaches

Distorted Probabilities and Multisets

1st approach: Definition based on a pseudoadditive integral:
Nondecreasing function-based fuzzy measures

- X a reference set, M a multiset on X and μ a $\oplus$-decomposable fuzzy measure on X. Let $f : [0, \infty) \to [0, \infty)$ be a non-decreasing function with $f(0) = 0$ and $f(m(M)) = 1$. Then, we define a fuzzy measure ν on $\mathcal{P}(M)$ by

$$\nu_f(A) = f(m(A))$$

where m is the multiset function $m : \mathcal{P}(M) \to [0, \infty)$ defined by

$$m(A) = (D) \int count_A d\mu.$$

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- Properties:
  if $A \subseteq B$ by the monotonicity of the integral $m(A) \leq m(B)$
  $\to$ monotonicity condition of the fuzzy measure fulfilled

Distorted Probabilities and Multisets

2nd approach: Definition based on prime numbers[1]:

- Define
$$n(A) := \prod_{x \in X} \phi(x)^{count_A(x)},$$
where ϕ is an injective function from X to the prime numbers, and let h be a non-decreasing function from $\mathbb{N}$ to $[0,1]$ satisfying $h(1) = 0$ and $h(n(M)) = 1$. We define the prime number-based fuzzy measure
$$\nu_{\phi,h}(A) = h(n(A)).$$

[1] and using the unique factorization of integers into prime numbers

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Properties:

if $A \neq B$ by the unique factorization $n(A) \neq n(B)$

if $A \subseteq B$ by the factorization $n(A) < n(B)$

$\rightarrow$ monotonicity condition of the fuzzy measure fulfilled

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- It seems that there is some parallelism between prime-number based fuzzy measures and distorted probabilities
  - f and the distortion
  - ϕ and the probability distribution
- Can we establish a relationship??

Results

Properties:

- ν a fuzzy measure according to Approach 1 on a proper finite set $(M, \mathcal{P}(M)) = (X, 2^X)$. Then ν is a distorted probability on $(X, 2^X)$.

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- ν a fuzzy measure according to Approach 1 on a proper finite set $(M, \mathcal{P}(M)) = (X, 2^X)$. Then ν is a distorted probability on $(X, 2^X)$.
- Same for Approach 2 (primer-number definition)
- This is easy to prove (consists on defining the probability distribution)
- So, Approach 1 and Approach 2 equal to or more general than distorted probabilities

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  $\rightarrow$ $\sum_{x \in A} p_x$ and $\prod_{x \in A} \phi(x)$ play the same role

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  Surprising corollary: Approach 1 and approach 2 are equivalent.

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  Surprising corollary: Approach 1 and approach 2 are equivalent.

  Proof based on some results on number theory about the existence of k prime numbers in certain intervals (Bertrand's postulate).

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An example to satisfy curiosity:

- μ distorted probability $p = (0.05, 0.1, 0.2, 0.3, 0.35)$, $g(x) = x^2$.

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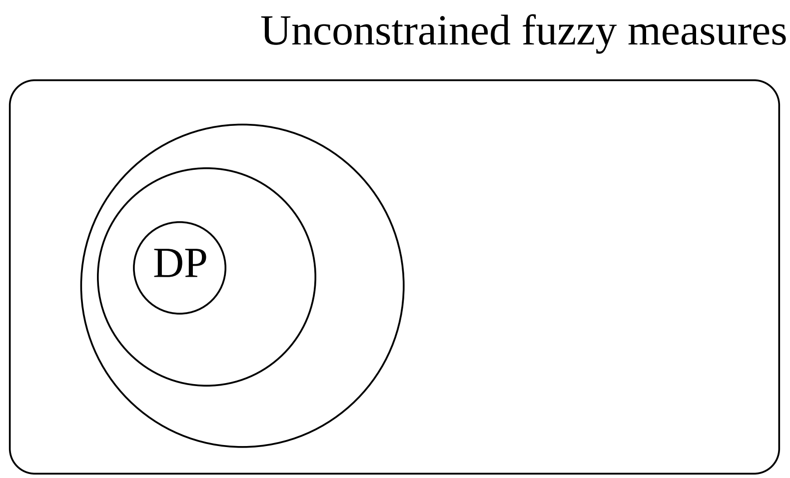
- μ distorted probability $p = (0.05, 0.1, 0.2, 0.3, 0.35)$, $g(x) = x^2$.
- Representation with prime numbers and appropriate function

$$
\begin{aligned}
\phi(x_1) &= 17 \in [16.0, 32.0001] \\
\phi(x_2) &= 367 \in [362.041, 724.081] \\
\phi(x_3) &= 185369 \in [185366.0, 370732.0] \\
\phi(x_4) &= 94907801 \in [9.49078 \times 10^7, 1.89816 \times 10^8] \\
\phi(x_5) &= 2147524151 \in [2.14752 \times 10^9, 4.29505 \times 10^9]
\end{aligned}
$$

m-Dimensional DP for multisets

How to solve the *problem* that not all fuzzy measures for multisets are distorted probabilities ?

- Same approach as before: m-dimensional prime number-based fuzzy measure



m-Dimensional DP for multisets

m-dimensional prime number-based fuzzy measure

- μ is an at most m-dimensional prime number-based fuzzy measure if

$$\mu(A) = f(n_1(A \cap X_1), \dots, n_m(A \cap X_m))$$

where,
$\{X_1, X_2, \cdots, X_m\}$ is a partition of X,
$n_i(A) = \prod_{x \in X_i} \phi(x)^{count_A(x)}$ with ϕ_i injective functions from X_i to the prime numbers
f is a strictly increasing function with respect to the i-th axis for all $i = 1, 2, \dots, m$.

μ is an m-dimensional prime number-based fuzzy measure if it is an at most m dimensional distorted probability but it is not an at most $m-1$ dimensional.

m-Dimensional DP for multisets

Properties:

- All fuzzy measures are at most $|X|$-dimensional prime number-based fuzzy measures.

Integral

Integral

Definition

- Boundary measures:
  - $\mu^+(A) = A \cdot M$ for all $A \subseteq X$
  - $\mu_-(A) = A \cap M$ for all $A \subseteq X$
- They satisfy:

$$\mu_-(A) \leq \mu^+(A)$$

and, therefore,

$$(C) \int f d\mu_- < (C) \int f d\mu^+$$

Finally an application

Record Linkage

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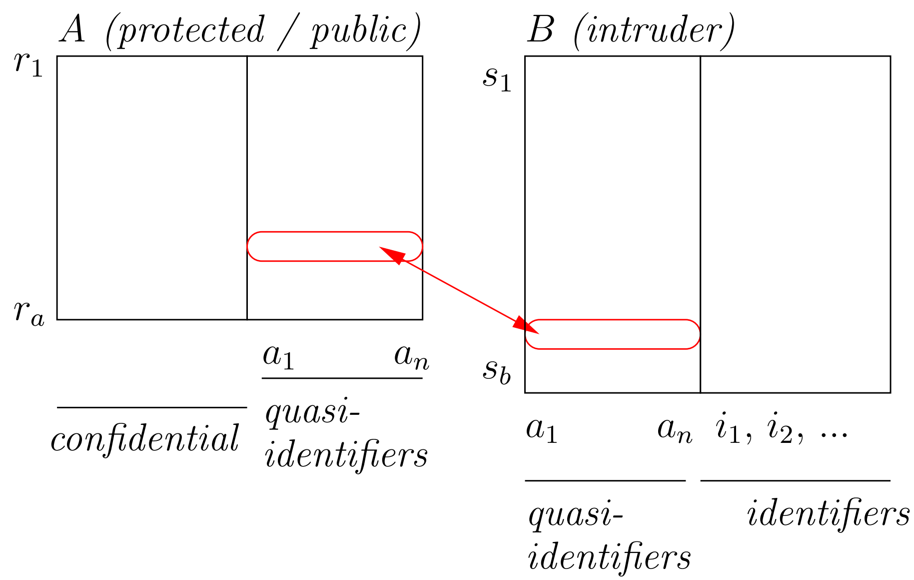

Record Linkage

Record Linkage:

$$Minimize \sum_{i=1}^{N} K_i \tag{1}$$

$$Subject\ to:$$

$$\sum_{i=1}^{N}\sum_{j=1}^{N} \mathbb{C}(d(V_1(a_i), V_1(b_j)), \ldots, d(V_n(a_i), V_n(b_j))) -$$

$$- \mathbb{C}(d(V_1(a_i), V_1(b_i)), \ldots, d(V_n(a_i), V_n(b_i))) + CK_i > 0 \tag{2}$$

$$K_i \in \{0, 1\} \tag{3}$$

Additional constraints according to $\mathbb{C}$ (4)

Summary

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Summary:

- Brief justification of the use of non-additive (fuzzy) measures
- Introduction to distorted probabilities
- Extensions
  - m-dimensional distorted probabilities
  - Fuzzy measures for multisets